Geometric theory of defects

M. O. Katanaev *

Steklov Mathematical Institute,
Gubkin St. 8
Moscow, 119991, Russia

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Abstract

A description of dislocations and disclinations defects in terms of Riemann–Cartan geometry is given, with the curvature and torsion tensors being interpreted as the surface densities of the Frank and Burgers vectors, respectively. A new free energy expression describing the static distribution of defects is presented, and equations of nonlinear elasticity theory are used to specify the coordinate system. Application of the Lorentz gauge leads to equations for the principal chiral $\mathbb{SO}(3)$-field. In the defect-free case, the geometric model reduces to elasticity theory for the displacement vector field and to a principal chiral $\mathbb{SO}(3)$-field model for the spin structure. As illustrated by the example of a wedge dislocation, elasticity theory reproduces only the linear approximation of the geometric theory of defects. It is shown that the equations of asymmetric elasticity theory for the Cosserat media can also be naturally incorporated into the geometric theory as the gauge conditions. As an application of the theory, phonon scattering on a wedge dislocation is considered. The energy spectrum of impurity in the field of a wedge dislocation is also discussed.

1 Introduction

Many solids have a crystalline structure. However, ideal crystals are absent in nature, and most of their physical properties, such as plasticity, melting, growth, etc., are defined by defects of the crystalline structure. Therefore, a study of defects is a topical scientific question of importance for applications in the first place. A broad experimental and theoretical investigations of defects in crystals started in the 1930s and continues nowadays. At present, a fundamental theory of defects is absent in spite of the existence of dozens of monographs and thousands of articles.

One of the most promising approaches to the theory of defects is based on Riemann–Cartan geometry, which involves nontrivial metric and torsion. In this approach, a crystal is considered as a continuous elastic medium with a spin structure. If the displacement

*E-mail: katanaev@mi.ras.ru

vector field is a smooth function, then there are only elastic stresses corresponding to diffeomorphisms of the Euclidean space. If the displacement vector field has discontinuities, then we are saying that there are defects in the elastic structure. Defects in the elastic structure are called dislocations and lead to the appearance of nontrivial geometry. Precisely, they correspond to a nonzero torsion tensor, equal to the surface density of the Burgers vector.

The idea to relate torsion to dislocations appeared in the 1950s [1–4]. This approach is still being successfully developed (note reviews [5–11]), and is often called the gauge theory of dislocations. A similar approach is also being developed in gravity [12]. It is interesting to note that E Cartan introduced torsion in geometry [13] having the analogy with mechanics of elastic media in mind.

The gauge approach to the theory of defects is being developed successfully, and interesting results are being obtained in this way [14–17]. We note in this connection two respects in which the approach proposed below is essentially different. In the gauge models of dislocations based on the translational group or on the semidirect product of the rotational group with translations, one usually chooses the distortion and displacement fields as independent variables. It is always possible to fix the invariance under local translations such that the displacement field becomes zero because it transforms by simple translation under the action of the translational group. In this sense, the displacement field is the gauge parameter of local translations, and physical observables are independent of it in gauge-invariant models.

The other disadvantage of the gauge approach is the equations of equilibrium. Einstein type equations are usually considered for distortion or vielbein, with the right hand side depending on the stress tensor. This appears unacceptable from the physical point of view because of the following reason. Consider, for example, one straight edge dislocation. In this case, the elastic stress field differs from zero everywhere. Then the torsion tensor (or curvature) is also nontrivial in the whole space due to the equations of equilibrium. This is wrong from our point of view. Indeed, we can consider an arbitrary domain of medium outside the cutting surface and look at the creation process for an edge dislocation. The chosen domain was a part of the Euclidean space with identically zero torsion and curvature before the defect creation. It is clear that torsion and curvature remain zero because the process of dislocation formation is a diffeomorphism for the considered domain. In addition, the cutting surface may be chosen arbitrary for the defect creation, leaving the dislocation axis unchanged. Then it follows that torsion and curvature must be zero everywhere except at the axis of dislocation. In other words, the elasticity stress tensor can not be the source of dislocations. To avoid the apparent contradiction, we propose a radical way out: we do not use the displacement field as an independent variable at all. This does not mean that the displacement field does not exist in real crystals. In the proposed approach, the displacement field exists and can be computed in those regions of medium that do not contain cores of dislocations. In this case, it satisfies the equations of nonlinear elasticity theory.

The proposed geometric approach allows considering other defects that do not relate directly to defects of elastic media.

The intensive investigations of other defects were conducted in parallel with the study of dislocations. The point is that many solids have not only elastic properties but also a spin structure. For example, there are ferromagnets, liquid crystals, spin glasses. In this

case, there are defects in the spin structure which are called disclinations [18]. They arise when the director field has discontinuities. The presence of disclinations is also connected to nontrivial geometry. Namely, the curvature tensor equals the surface density of the Frank vector. The gauge approach based on the rotational group $\mathbb{SO}(3)$ was also used for describing disclinations [19]. $\mathbb{SO}(3)$-gauge models of spin glasses with defects were considered in [20, 21].

The geometric theory of static distribution of defects which describes both types of defects – dislocation and disclinations – from a single standpoint was proposed in [22]. In contrast to other approaches, it involves the vielbein and $\mathbb{SO}(3)$ connection as the only independent variables. The torsion and curvature tensors have direct physical meaning as the surface densities of dislocations and disclinations, respectively. Covariant equations of equilibrium for the vielbein and $\mathbb{SO}(3)$ connection similar to those in a gravity model with torsion are postulated. To define the solution uniquely, we must fix the coordinate system (fix the gauge) because any solution of the equations of equilibrium is defined up to general coordinate transformations and local $\mathbb{SO}(3)$ rotations. The elastic gauge for the vielbein [23] and Lorentz gauge for the $\mathbb{SO}(3)$ connection [24] were proposed recently. We stress that the notions of a displacement vector and rotational angle are completely absent in our approach. These notions can be introduced only in those domains where defects are absent. In this case, equations for vielbein and $\mathbb{SO}(3)$ connection are identically satisfied, the elastic gauge reduces to the equations of nonlinear elasticity theory for the displacement vector, and the Lorentz gauge leads to the equations for the principal chiral $\mathbb{SO}(3)$ field. In other words, to fix the coordinate system, we choose two fundamental models: the elasticity theory and the principal chiral field model.

To show the advantages of the geometric approach and to compare it with the elasticity theory, we consider in detail a wedge dislocation in the frameworks of the elasticity theory and the proposed geometric model. We show that the explicit expression for the metric in the geometric approach is simpler and coincides with the induced metric obtained within the elasticity theory only for small relative deformations.

As an application of the geometric theory of defects, we consider two examples in the last sections of the present review. First, we solve the problem of phonon scattering on a wedge dislocation. The problem of phonon scattering is reduced to the integration of equations for extremals for the metric describing a wedge dislocation (because phonons move along extremals in the eikonal approximation). As a second application, we consider the quantum mechanical problem of impurity or vacancy motion inside a cylinder whose axis coincides with a wedge dislocation. The wave functions and energetic spectrum of the impurity are found explicitly.

The presence of defects results in nontrivial Riemann–Cartan geometry. This means that for describing the phenomena that relate ingeniously to elastic media, we must make changes in the corresponding equations. For example, if a phonon propagation in an ideal crystal is described by the wave equation, then the presence of defects is easily taken into account. For this, the flat Euclidean metric has to be replaced by a nontrivial metric describing the distribution of defects. The same substitution must be made in the Schrödinger equation to describe other quantum effects. It is shown nowadays that the presence of defects essentially influences physical phenomena. The Schrödinger equation in the presence of dislocations was considered in [25–46] for different problems. Problems related to the wave or Laplace equations were considered in [47–54]. The influence of the

nontrivial metric related to the presence of defects was investigated in electrodynamics [55] and hydrodynamics [56]. Scattering of phonons on straight parallel dislocations was studied in [57–59].

Another approach to the theory of defects based on affine geometry with nonzero nonmetricity tensor was considered recently in [60].

2 Elastic deformations

We consider infinite three dimensional elastic media. We suppose that the undeformed medium in the defect-free case is invariant under translations and rotations in some coordinate system. Then, the medium in this coordinate system y^i, $i = 1, 2, 3$, is described by the Euclidean metric $\delta_{ij} = \mathsf{diag}\,(+++)$, and the system of coordinates is called Cartesian. Thus, in the undeformed state, we have the Euclidean space $\mathbb{R}^3$ with a given Cartesian coordinate system. We also assume that torsion (see the Appendix) in the medium equals zero.

Let a point of the medium has coordinates y^i in the ground state. After deformation, this point has the coordinates

$$y^i \to x^i(y) = y^i + u^i(x) \tag{1}$$

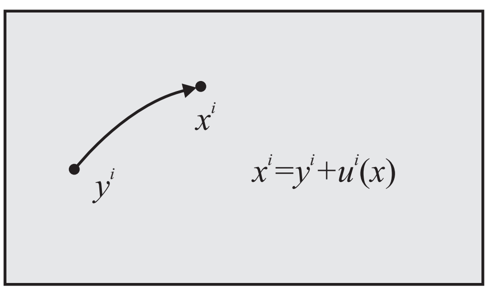

Figure 1: Elastic deformations

in the initial coordinate system, see Fig. 1. The inverse notation is used in the elasticity theory. One usually writes $x^i \to y^i = x^i + u^i(x)$. These are equivalent because both coordinate systems x^i and y^i cover the whole $\mathbb{R}^3$. However, in the theory of defects considered in the next sections, the situation is different. Generally, the elastic medium fills the whole Euclidean space only in the final state. Here and in what follows, we assume that fields depend on coordinates x that are coordinates of points of the medium after the deformation and cover the whole Euclidean space $\mathbb{R}^3$. In the presence of dislocations, the coordinates y^i do not cover the whole $\mathbb{R}^3$ in the general case because part of the media may be removed or, conversely, added. Therefore, the system of coordinates related to points of the medium after an elastic deformation and defect creation is more preferable.

In the linear elasticity theory, relative deformations are assumed to be small ($\partial_j u^i \ll 1$). The functions $u^i(x) = u^i(y(x))$ are then components of a vector field that is called the displacement vector field and is the basic variable in elasticity theory.

In the absence of defects, we assume that the displacement field is a smooth vector field in the Euclidean space $\mathbb{R}^3$. The presence of discontinuities and singularities of the displacement field is interpreted as a presence of defects in elastic media.

In what follows, we consider only static deformations with the displacement field u^i independent of time. Then the basic equations of equilibrium for small deformations are

(see, e.g., [61])

$$\partial_j \sigma^{ji} + f^i = 0, \tag{2}$$

$$\sigma^{ij} = \lambda \delta^{ij} \epsilon_k{}^k + 2\mu \epsilon^{ij}, \tag{3}$$

where σ^{ij} is the stress tensor, which is assumed to be symmetric. The tensor of small deformations ϵ_{ij} is given by the symmetrized partial derivative of the displacement vector:

$$\epsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i). \tag{4}$$

Lowering and raising of the Latin indices is performed with the Euclidean metric δ_{ij} and its inverse δ^{ij}. The letters λ and μ denote constants characterizing elastic properties of media and are called Lamé coefficients. Functions $f^i(x)$ describe the total density of nonelastic forces inside the medium. We assume in what follows that such forces are absent: $f^j(x) = 0$. Equation (2) is Newton's law, and Eqn (3) is Hook's law relating stresses to deformations.

In a Cartesian coordinate system and for small deformations, the difference between upper and lower indices disappears because raising and lowering of indices is performed with the help of the Euclidean metric. One usually forgets about this difference due to this reason, and this is fully justified. But in the presence of defects, the notion of Cartesian coordinate system and Euclidean metric is absent, and the indices are raised and lowered with the help of Riemannian metric. Therefore, we distinguish the upper and lower indices as is accepted in differential geometry, having the transition to elastic media with defects in mind.

The main problem in the linear elasticity theory is the solution of the second-order equations for the displacement vector that arise after substitution of (3) into (2) with some boundary conditions. Many known solutions are in good agreement with experiment. Therefore, one may say that equations (3), (2) have a solid experimental background.

We now look at the elastic deformations from the standpoint of differential geometry. From the mathematical standpoint, map (1) is itself a diffeomorphism of the Euclidean space $\mathbb{R}^3$. The Euclidean metric δ_{ij} is then induced by the map $y^i \to x^i$. It means that in the deformed state, the metric in the linear approximation is given by

$$g_{ij}(x) = \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} \delta_{kl} \approx \delta_{ij} - \partial_i u_j - \partial_j u_i = \delta_{ij} - 2\epsilon_{ij}, \tag{5}$$

i.e., is defined by the tensor of small deformations (4). We note that in the linear approximation, $\epsilon_{ij}(x) = \epsilon_{ij}(y)$ and $\partial u_j / \partial x^i = \partial u_j / \partial y^i$.

In Riemannian geometry, the metric uniquely defines the Levi–Civita connection $\widetilde{\Gamma}_{ij}{}^k(x)$ (Christoffel's symbols), Eqn (115). We can compute curvature tensor (119) for these symbols. This tensor equals identically zero, $\widetilde{R}_{ijk}{}^l(x) = 0$, because the curvature of the Euclidean space is zero, and the map $y^i \to x^i$ is a diffeomorphism. The torsion tensor is equal to zero for the same reason. Thus, an elastic deformation of the medium corresponds to the trivial Riemann–Cartan geometry, with zero curvature and torsion tensors.

The physical interpretation of metric (5) is as follows. The external observer fixes Cartesian coordinate system corresponding to the ground undeformed state of the medium.

The medium is then deformed, and external observer discovers that the metric becomes nontrivial in this coordinate system. If we assume that elastic perturbations in the medium (phonons) propagate along extremals (lines of minimal length), then their trajectories in the deformed medium are defined by Eqns (118). Trajectories of phonons are now not straight lines because the Christoffel's symbols are nontrivial ($\widetilde{\Gamma}_{jk}{}^i \neq 0$). In this sense, metric (5) is observable. Here, we see the essential role of the Cartesian coordinate system y^i defined by the undeformed state, with which the measurement process is connected.

We assume that the metric $g_{ij}(x)$ given in the Cartesian coordinates corresponds to some state of elastic media without defects. The displacement vector is then defined by the system of equations (5), and its integrability conditions are the equality of the curvature tensor to zero, in accordance with theorem 2 in the Appendix. In the linear approximation, these conditions are known in elasticity theory as the Saint–Venant integrability conditions.

We make a remark that is important for the following consideration. For appropriate boundary conditions, the solution of the elasticity theory equations (2), (3) is unique. From the geometric standpoint, this means that elasticity theory fixes diffeomorphisms. This fact is used in the geometric theory of defects. Equations of nonlinear elasticity theory written in terms of the metric or vielbein are used for fixing the coordinate system.

3 Dislocations

We start with the description of linear dislocations in elastic media (see, e.g., [61, 62]). The simplest and most undespread examples of linear dislocations are shown in Fig. 2. We cut the medium along the half-plane $x^2 = 0$, $x^1 > 0$, move the upper part of the medium located over the cut $x^2 > 0$, $x^1 > 0$ by the vector $\boldsymbol{b}$ towards the dislocation axis x^3, and glue the cutting surfaces. The vector $\boldsymbol{b}$ is called the Burgers vector. In the general case, the Burgers vector may not be constant on the cut. For the edge dislocation, it varies from zero to some constant value $\boldsymbol{b}$ as it moves from the dislocation axis. After the gluing, the media comes to the equilibrium state called the edge dislocation, see Fig. 2*a*. If the Burgers vector is parallel to the dislocation line, it is called the screw dislocation (Fig. 2*b*).

The same dislocation can be made in different ways. For example, if the Burgers vector is perpendicular to the cutting plane and directed from it in the considered cases, then the produced cavity must be filled with medium before gluing. It is easy to imagine that the edge dislocation is also obtained as a result, but rotated by the angle $\pi/2$ around the x^3 axis. This example shows that a dislocation is characterized not by the cutting surface but by the dislocation line and the Burgers vector.

From the topological standpoint, the medium containing several dislocations or even the infinite number of them is the Euclidean space $\mathbb{R}^3$. In contrast to the case of elastic deformations, the displacement vector in the presence of dislocations is no longer a smooth function because of the presence of cutting surfaces. At the same time, we assume that partial derivatives of the displacement vector $\partial_j u^i$ (the distortion tensor) are smooth functions on the cutting surface. This assumption is justified physically because these derivatives define deformation tensor (4). In its turn, partial derivatives of the deformation tensor must exist and be smooth functions in the equilibrium state everywhere except,

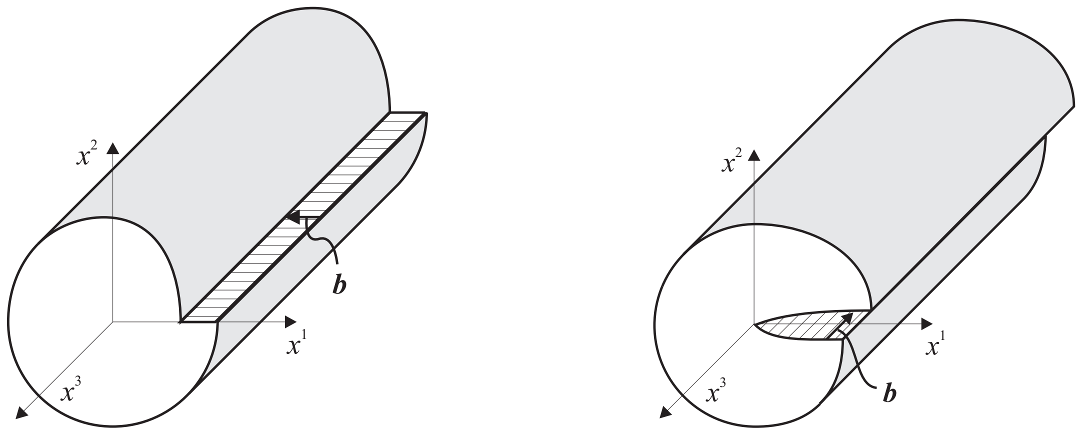

Figure 2: Straight linear dislocations. (a) The edge dislocation. The Burgers vector $\boldsymbol{b}$ is perpendicular to the dislocation line. (b) The screw dislocation. The Burgers vector $\boldsymbol{b}$ is parallel to the dislocation line.

possibly, the dislocation axis, because otherwise equations of equilibrium (2) have no meaning. We assume that the metric and vielbein are smooth functions everywhere in $\mathbb{R}^3$ except, may be, dislocation axes, because the deformation tensor defines the induced metric (5).

The main idea of the geometric approach amounts to the following. To describe single dislocations in the framework of elasticity theory we must solve equations for the displacement vector with some boundary conditions on the cuts. This is possible for small number of dislocations. But, with an increasing number of dislocations, the boundary conditions become so complicated that the solution of the problem becomes unrealistic. Besides, one and the same dislocation can be created by different cuts which leads to an ambiguity in the displacement vector field. Another shortcoming of this approach is that it cannot be applied to the description of a continuous distribution of dislocations because the displacement vector field does not exist in this case at all because it must have discontinuities at every point. In the geometric approach, the basic variable is the vielbein which by assumption is a smooth function everywhere except, possibly, dislocation axes. We postulate new equations for the vielbein (see section 5). In the geometric approach, the transition from a finite number of dislocations to their continuous distribution is simple and natural. In that way, the smoothing of singularities occurs on dislocation axes in analogy with smoothing of mass distribution for point particles in passing to continuous media.

We now develop the formalism of the geometric approach. In a general defect-present case, we do not have a preferred Cartesian coordinate frame in the equilibrium because there is no symmetry. Therefore, we consider arbitrary coordinates x^μ, $\mu = 1, 2, 3$, in $\mathbb{R}^3$. We use Greek letters for coordinates allowing arbitrary coordinate changes. Then the Burgers vector can be expressed as the integral of the displacement vector

$$\oint_C dx^\mu \partial_\mu u^i(x) = -\oint_C dx^\mu \partial_\mu y^i(x) = -b^i, \tag{6}$$

where C is a closed contour surrounding the dislocation axis, Fig. 3.

This integral is invariant under arbitrary coordinate transformations $x^\mu \to x^{\mu'}(x)$ and covariant under global $\mathbb{SO}(3)$-rotations of y^i. Here, components of the displacement vector field $u^i(x)$ are considered with respect to the orthonormal basis in the tangent space, $u = u^i e_i$. If components of the displacement vector field are considered with respect to the coordinate basis $u = u^\mu \partial_\mu$, the invariance of the integral (6) under general coordinate changes is violated.

In the geometric approach, we introduce new independent variable – the vielbein – instead of partial derivatives $\partial_\mu u^i$

$$e_\mu{}^i(x) = \begin{cases} \partial_\mu y^i, & \text{outside the cut,} \\ \lim \partial_\mu y^i, & \text{on the cut.} \end{cases} \tag{7}$$

The vielbein is a smooth function on the cut by construction. We note that if the vielbein was simply defined as the partial derivative $\partial_\mu y^i$, then it would have the δ-function singularity on the cut because functions $y^i(x)$ have a jump. The Burgers vector can be expressed through the integral over a surface S having contour C as the boundary

$$\oint_C dx^\mu e_\mu{}^i = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i) = b^i, \tag{8}$$

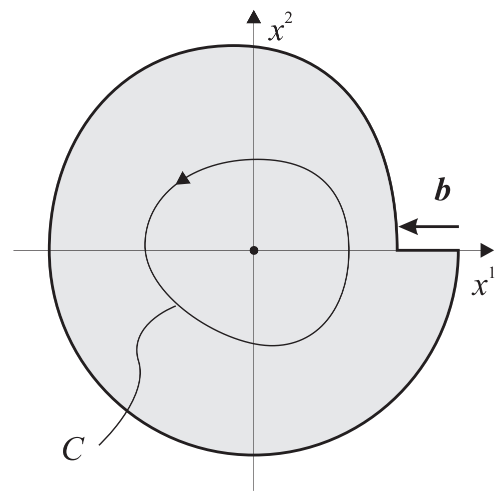

Figure 3: Section of the media with the edge dislocation. C is the integration contour for the Burgers vector $\boldsymbol{b}$.

where $dx^\mu \wedge dx^\nu$ is the surface element. As a consequence of the definition of the vielbein in (7), the integrand is equal to zero everywhere except at the dislocation axis. For the edge dislocation with constant Burgers vector, the integrand has a δ-function singularity at the origin. The criterion for the presence of a dislocation is a violation of the integrability conditions for the system of equations $\partial_\mu y^i = e_\mu{}^i$:

$$\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i \neq 0. \tag{9}$$

If dislocations are absent, then the functions $y^i(x)$ exist and define transformation to a Cartesian coordinates frame.

In the geometric theory of defects, the field $e_\mu{}^i$ is identified with the vielbein. Next, we compare the integrand in (8) with the expression for the torsion in Cartan variables (125). They differ only by terms containing the $\mathbb{SO}(3)$ connection. This is the ground for the introduction of the following postulate. In the geometric theory of defects, the Burgers vector corresponding to a surface S is defined by the integral of the torsion tensor:

$$b^i = \iint_S dx^\mu \wedge dx^\nu T_{\mu\nu}{}^i.$$

This definition is invariant with respect to general coordinate transformations of x^μ and covariant with respect to global rotations. Thus, the torsion tensor has straightforward physical interpretation: it is equal to the surface density of the Burgers vector.

The physical interpretation of the $\mathbb{SO}(3)$ connection is given in section 4, and now we show how this definition reduces to the expression for the Burgers vector (8) obtained within elasticity theory. If the curvature tensor for the $\mathbb{SO}(3)$ connection is zero, then, according to theorem 3, the connection is locally trivial, and there exists such $\mathbb{SO}(3)$ rotation such that $\omega_{\mu i}{}^{j} = 0$. In this case, we return to expression (8).

If the $\mathbb{SO}(3)$ connection is zero and vielbein is a smooth function, then the Burgers vector corresponds uniquely to every contour. It can then be expressed as a surface integral of the torsion tensor. The surface integral depends only on the boundary contour but not on the surface due to the Stokes theorem.

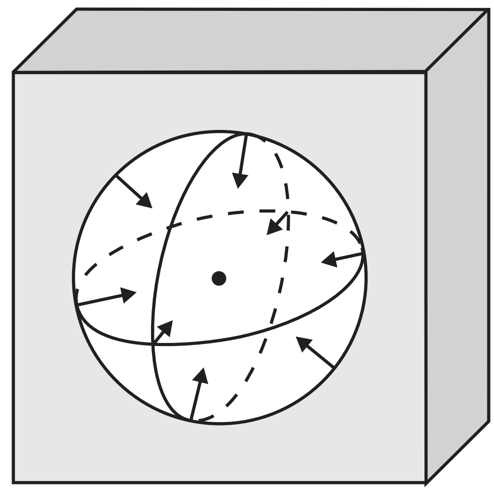

Figure 4: Point defect: a vacancy appears when a ball is cut out from the medium, and the boundary sphere is shrunk to a point.

We have shown that the presence of linear defects results in a nontrivial torsion tensor. In the geometric theory of defects, the equality of the torsion tensor to zero $T_{\mu\nu}{}^{i} = 0$ is naturally considered the criterion for the absence of dislocations. Then, under the name dislocation fall not only linear dislocations but, in fact, arbitrary defects in elastic media. For example, point defects; vacancies and impurities, are also dislocations. In the first case we cut out a ball from the Euclidean space $\mathbb{R}^3$ and then shrink the boundary sphere to a point (Fig. 4). In the case of impurity, a point of the Euclidean space is blown up to a sphere and the produced cavity is filled with the medium. Point defects are characterized by the mass of the removed or added media, which is also defined by the vielbein [22]

$$M = \rho_0 \iiint_{\mathbb{R}^3} d^3x \left(\det e_\mu{}^i - \det \overset{\circ}{e}_\mu{}^i \right), \qquad \overset{\circ}{e}_\mu{}^i = \partial_\mu y^i, \tag{10}$$

where $y^i(x)$ are the transition functions to a Cartesian coordinate frame in $\mathbb{R}^3$ and ρ_0 is the density of the medium which is supposed to be constant. The mass is defined by the difference of two integrals, each of them being separately divergent. The first integral equals to the volume of the medium with defects and the second is equal to the volume of the Euclidean space. The torsion tensor for a vacancy or impurity is zero everywhere except at one point, where it has a δ-function singularity. For point defects, the notion of the Burgers vector is absent.

According to the given definition, the mass of an impurity is positive because the matter is added to the media, and the mass of a vacancy is negative because part of the medium is removed. The negative sign of the mass causes serious problems for a physical interpretation of solutions of the equations of motion or the Schrödinger equation. Hence, we make a remark. Strictly speaking, the integral (10) should be called the "bare" mass because this expression does not account for elastic stresses arising around a point dislocation. The effective mass of such a defect must contain at least two contributions: the bare mass and the free energy coming from elastic stresses. The question about the sign of the effective mass is not solved yet and demands a separate analysis.

Surface defects may also exist in three-dimensional space, in addition to point and line

dislocations. In the geometric approach, all of them are called dislocations because they correspond to a nontrivial torsion.

4 Disclinations

In the preceeding section, we related dislocations to a nontrivial torsion tensor. For this, we introduced an $\mathbb{SO}(3)$ connection. Now we show that the curvature tensor for the $\mathbb{SO}(3)$ connection defines the surface density of the Frank vector characterizing other well-known defects – disclinations in the spin structure of media [61].

Let a unit vector field $n^i(x)$ $(n^i n_i = 1)$ be given at all points of the medium. For example, n^i has the meaning of the magnetic moment located at each point of the medium for ferromagnets (Fig. 5a). For nematic liquid crystals, the unit vector field n^i with the equivalence relation $n^i \sim -n^i$ describes the director field (Fig. 5b).

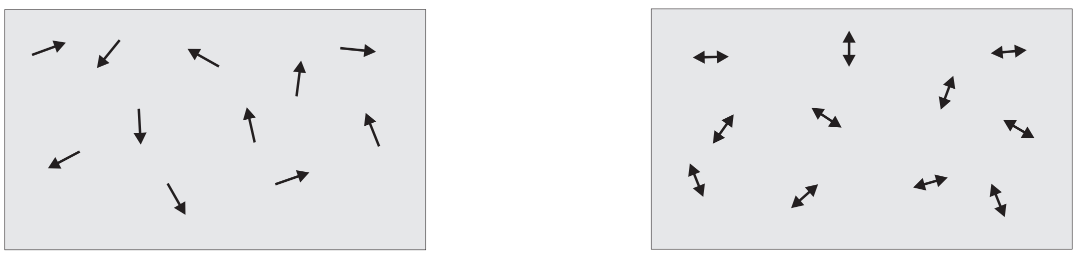

Figure 5: Examples of media with the spin structure: (a) ferromagnet, (b) liquid crystal.

We fix some direction in the medium n^i_0. Then the field $n^i(x)$ at a point x can be uniquely defined by the field $\omega^{ij}(x) = -\omega^{ji}(x)$ taking values in the rotation algebra $\mathfrak{so}(3)$ (the rotation angle),

$$n^i = n^j_0 S_j{}^i(\omega),$$

where $S_j{}^i \in \mathbb{SO}(3)$ is the rotation matrix corresponding to the algebra element ω^{ij}. Here, we use the following parameterization of the rotation group $\mathbb{SO}(3)$ by elements of its algebra (see, e.g., [24])

$$S_i{}^j = (e^{(\omega\varepsilon)})_i{}^j = \cos\omega\,\delta^j_i + \frac{(\omega\varepsilon)_i{}^j}{\omega}\sin\omega + \frac{\omega_i\omega^j}{\omega^2}(1-\cos\omega) \in \mathbb{SO}(3), \qquad (11)$$

where $(\omega\varepsilon)_i{}^j = \omega^k\varepsilon_{ki}{}^j$ and $\omega = \sqrt{\omega^i\omega_i}$ is the modulus of the vector ω^i. The pseudovector $\omega^k = \omega_{ij}\varepsilon^{ijk}/2$, where ε^{ijk} is the totally antisymmetric third-rank tensor, $\varepsilon^{123} = 1$, is directed along the rotation axis and its length equals the rotation angle. We call the field ω^{ij} spin structure of the media.

If a media has a spin structure, then it may have defects called disclinations. For linear disclinations parallel to the x^3 axis, the vector field n lies in the perpendicular plain x^1, x^2. The simplest examples of linear disclinations are shown in Fig. 6. Every linear disclination is characterized by the Frank vector

$$\Theta_i = \epsilon_{ijk}\Omega^{jk}, \qquad (12)$$

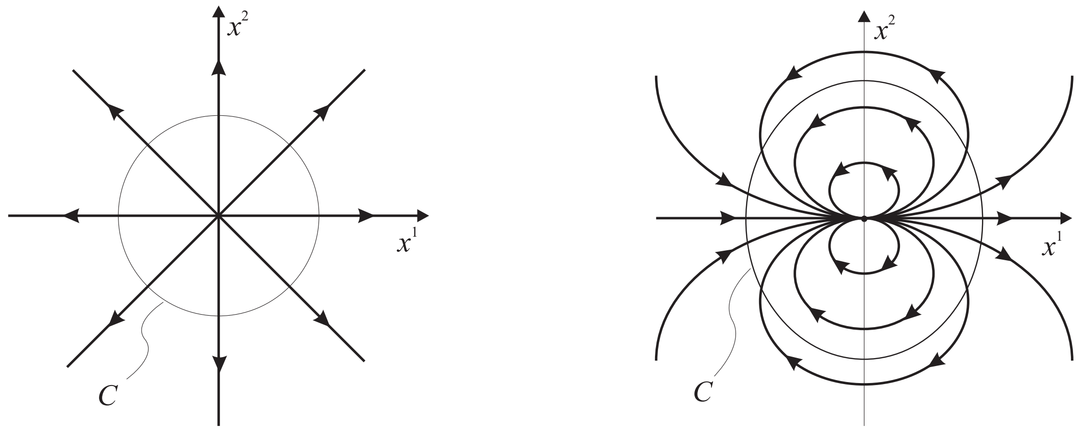

Figure 6: The vector field distributions on the plane x^1, x^2 for the linear disclinations parallel to the x^3 axis. (a) $|\Theta| = 2\pi$. (b) $|\Theta| = 4\pi$.

where

$$\Omega^{ij} = \oint_C dx^\mu \partial_\mu \omega^{ij}, \tag{13}$$

and the integral is taken along closed contour C surrounding the disclination axis. The length of the Frank vector is equal to the total angle of rotation of the field n^i as it goes around the disclination.

The vector field n^i defines a map of the Euclidean space to a sphere $n:\ \mathbb{R}^3 \to \mathbb{S}^2$. For linear disclinations parallel to the x^3 axis, this map is restricted to a map of the plane $\mathbb{R}^2$ to a circle $\mathbb{S}^1$. In this case, the total rotation angle must obviously be a multiple of 2π.

For nematic liquid crystals, we have the equivalence relation $n^i \sim -n^i$. Therefore, for linear disclinations parallel to the x^3 axis, the director field defines a map of the plane into the projective line $n:\ \mathbb{R}^2 \to \mathbb{RP}^1$. In this case, the length of the Frank vector must be a multiple of π. The corresponding examples of disclinations are shown in Fig. 7.

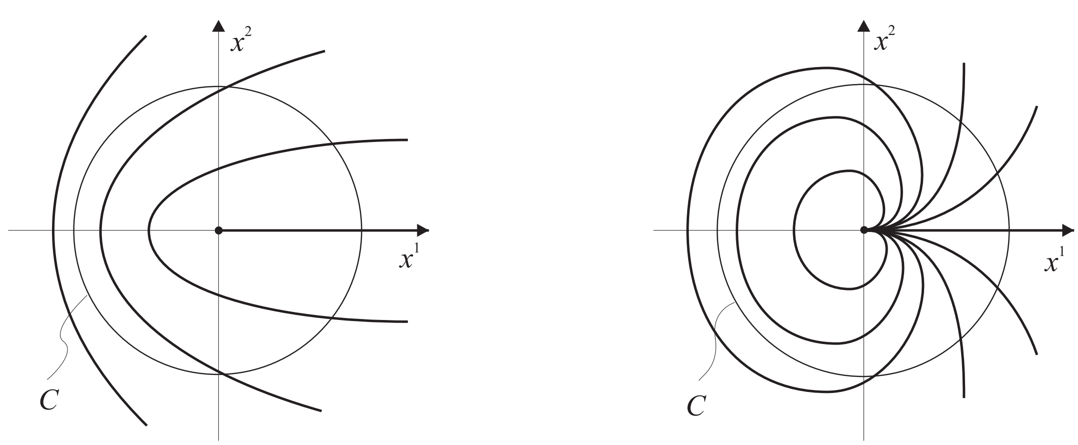

Figure 7: The director field distribution in the x^1, x^2 plane for the linear disclinations parallel to the x^3 axis. (a) $|\Theta| = \pi$ and (b) $|\Theta| = 3\pi$.

As for the displacement field, the field $\omega^{ij}(x)$, taking values in the algebra $\mathfrak{so}(3)$, is not a smooth function in $\mathbb{R}^3$ in the presence of disclinations. We make a cut in $\mathbb{R}^3$ bounded by the disclination axis. Then the field $\omega^{ij}(x)$ may be considered smooth in the whole space except the cut. We assume that all partial derivatives of $\omega^{ij}(x)$ have the same limit as it approaches the cut from both sides. Then we define the new field

$$\omega_\mu{}^{ij} = \begin{cases} \partial_\mu \omega^{ij}, & \text{outside the cut}, \\ \lim \partial_\mu \omega^{ij}, & \text{on the cut}. \end{cases} \tag{14}$$

The functions $\omega_\mu{}^{ij}$ are smooth everywhere by construction except, may be, on the disclination axis. Then the Frank vector may be expressed as the surface integral

$$\Omega^{ij} = \oint_C dx^\mu \omega_\mu{}^{ij} = \iint_S dx^\mu \wedge dx^\nu (\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij}), \tag{15}$$

where S is an arbitrary surface having the contour C as the boundary. If the field $\omega_\mu{}^{ij}$ is given, then the integrability conditions for the system of equations $\partial_\mu \omega^{ij} = \omega_\mu{}^{ij}$ are

$$\partial_\mu \omega_\nu{}^{ij} - \partial_\nu \omega_\mu{}^{ij} = 0. \tag{16}$$

This noncovariant equality yields the criterion for the absence of disclinations.

In the geometric theory of defects, we identify the field $\omega_\mu{}^{ij}$ with the $\mathbb{SO}(3)$ connection. In the expression for the curvature in (126), the first two terms coincide with (16), and we therefore postulate the covariant criterion of the absence of disclinations as the equality of the curvature tensor for the $\mathbb{SO}(3)$ connection to zero:

$$R_{\mu\nu}{}^{ij} = 0.$$

Simultaneously, we give the physical interpretation of the curvature tensor as the surface density of the Frank vector

$$\Omega^{ij} = \iint dx^\mu \wedge dx^\nu R_{\mu\nu}{}^{ij}. \tag{17}$$

This definition reduces to the previous expression for the Frank vector (15) in the case where rotation of the vector n occures in a fixed plane. In this case, rotations are restricted by the subgroup $\mathbb{SO}(2) \subset \mathbb{SO}(3)$. The quadratic terms in the expression for the curvature in (126) disappear because the rotation group $\mathbb{SO}(2)$ is Abelian, and we obtain the previous expression for the Frank vector (15).

Thus, we described the media with dislocations (defects of elastic media) and disclinations (defects in the spin structure) in the framework of Riemann–Cartan geometry, the torsion and curvature tensors being identified with the surface density of dislocations and disclinations, respectively. The relations between physical and geometrical notions are summarized in the Table 1.

The same physical interpretation of torsion and curvature was considered in [63]. Several possible functionals for the free energy were also considered. In the next section, we propose a new expression for the free energy.

Existence of defects	$R_{\mu\nu}{}^{ij}$	$T_{\mu\nu}{}^{i}$
Elastic deformations	0	0
Dislocations	0	$\neq 0$
Disclinations	$\neq 0$	0
Dislocations and disclinations	$\neq 0$	$\neq 0$

Table 1: The relation between physical and geometrical notions in the geometric theory of defects.

5 Free energy

Until now, we discussed only the relation between physical and geometrical notions. To complete the construction of the geometric theory of defects, we have to postulate equations of equilibrium describing static distribution of defects in media. The vielbein $e_\mu{}^i$ and $\mathbb{SO}(3)$ connection $\omega_\mu{}^{ij}$ are basic and independent variables in the geometric approach. In contrast to previous geometric approaches, we completely abandon the displacement field u^i and spin structure ω^{ij} as the fields entering the system of equilibrium equations. In a general case of a continuous distribution of defects, they simply do not exist. Nevertheless, at some level and under definite circumstances they can be reconstructed, and we discuss this in the following section.

The expression for the free energy was derived in [22]. We assume that equations of equilibrium must be covariant under general coordinate transformations and local rotations, be at most of the second order, and follow from a variational principle. The expression for the free energy leading to the equilibrium equations must then be equal to a volume integral of the scalar function (the Lagrangian) that is quadratic in torsion and curvature tensors. There are three independent invariants quadratic in the torsion tensor and three independent invariants quadratic in the curvature tensor in three dimensions. It is possible to add the scalar curvature and a "cosmological" constant Λ. We thus obtain a general eight-parameter Lagrangian

$$\begin{aligned}\frac{1}{e}L = &-\kappa R + \frac{1}{4}T_{ijk}(\beta_1 T^{ijk} + \beta_2 T^{kij} + \beta_3 T^j \delta^{ik}) \\ &+ \frac{1}{4}R_{ijkl}(\gamma_1 R^{ijkl} + \gamma_2 R^{klij} + \gamma_3 R^{ik}\delta^{jl}) - \Lambda, \qquad e = \det e_\mu{}^i,\end{aligned} \tag{18}$$

where κ, $\beta_{1,2,3}$ and $\gamma_{1,2,3}$ are some constants, and we have introduced the trace of torsion tensor $T_j = T_{ij}{}^i$ and the Ricci tensor $R_{ik} = R_{ijk}{}^j$. Here and in what follows, transformation of the Greek indices into the Latin ones and vice versa is always performed using the vielbein and its inverse. For example,

$$R_{ijkl} = R_{\mu\nu kl} e^\mu{}_i e^\nu{}_j, \qquad T_{ijk} = T_{\mu\nu k} e^\mu{}_i e^\nu{}_j.$$

The particular feature of three dimensions is that the full curvature tensor is in a one-to-one correspondence with Ricci tensor (129) and has three irreducible components. Therefore, the Lagrangian contains only three independent invariants quadratic in curvature tensor. We do not need to add the Hilbert–Einstein Lagrangian $\widetilde{R}$, also yielding second-order equations, to the free energy (18) because of identity (130).

Thus, the most general Lagrangian depends on eight constants and leads to very complicated equations of equilibrium. At present, we do not know precisely what values of the constants describe this or that medium. Therefore, we make physically reasonable assumptions to simplify matters. Namely, we require that equations of equilibrium must admit the following three types of solutions.

1. There are solutions describing the media with only dislocations,
   $R_{\mu\nu}{}^{ij} = 0,\ T_{\mu\nu}{}^{i} \neq 0.$

2. There are solutions describing the media with only disclinations,
   $R_{\mu\nu}{}^{ij} \neq 0,\ T_{\mu\nu}{}^{i} = 0.$

3. There are solutions describing the media without dislocations and disclinations,
   $R_{\mu\nu}{}^{ij} = 0,\ T_{\mu\nu}{}^{i} = 0.$

It turns out that these simple assumptions reduce the number of independent parameters from eight to two. We now prove this statement. The Lagrangian (18) yields the equilibrium equations

$$
\begin{aligned}
\frac{1}{e}\frac{\delta L}{\delta e_\mu{}^i} &= \kappa\left(Re^\mu{}_i - 2R_i{}^\mu\right) + \beta_1\left(\nabla_\nu T^{\nu\mu}{}_i - \frac{1}{4}T_{jkl}T^{jkl}e^\mu{}_i + T^{\mu jk}T_{ijk}\right) \\
&+ \beta_2\left(-\frac{1}{2}\nabla_\nu(T_i{}^{\mu\nu} - T_i{}^{\nu\mu}) - \frac{1}{4}T_{jkl}T^{ljk}e^\mu{}_i - \frac{1}{2}T^{j\mu k}T_{kij} + \frac{1}{2}T^{jk\mu}T_{kij}\right) \\
&+ \beta_3\left(-\frac{1}{2}\nabla_\nu(T^\nu e^\mu{}_i - T^\mu e^\nu{}_i) - \frac{1}{4}T_jT^je^\mu{}_i + \frac{1}{2}T^\mu T_i + \frac{1}{2}T^jT_{ij}{}^\mu\right) \\
&+ \gamma_1\left(-\frac{1}{4}R_{jklm}R^{jklm}e^\mu{}_i + R^{\mu jkl}R_{ijkl}\right) \\
&+ \gamma_2\left(-\frac{1}{4}R_{jklm}R^{lmjk}e^\mu{}_i + R^{kl\mu j}R_{ijkl}\right) \\
&+ \gamma_3\left(-\frac{1}{4}R_{jk}R^{jk}e^\mu{}_i + \frac{1}{2}R^{\mu j}R_{ij} + \frac{1}{2}R^{jk}R_{jik}{}^\mu\right) + \Lambda e^\mu{}_i = 0, \qquad (19) \\
\frac{1}{e}\frac{\delta L}{\delta\omega_\mu{}^{ij}} &= \kappa\left(\frac{1}{2}T_{ij}{}^\mu + T_ie^\mu{}_j\right) + \beta_1\frac{1}{2}T^\mu{}_{ji} + \beta_2\frac{1}{4}\left(T_i{}^\mu{}_j - T_{ij}{}^\mu\right) \\
&+ \beta_3\frac{1}{4}T_je^\mu{}_i + \gamma_1\frac{1}{2}\nabla_\nu R^{\nu\mu}{}_{ij} + \gamma_2\frac{1}{2}\nabla_\nu R_{ij}{}^{\nu\mu} \\
&+ \gamma_3\frac{1}{4}\nabla_\nu\left(R^\nu{}_ie^\mu{}_j - R^\mu{}_ie^\nu{}_j\right) - (i\leftrightarrow j) = 0, \qquad (20)
\end{aligned}
$$

where the covariant derivative acts with the $\mathbb{SO}(3)$ connection on the Latin indices and with the Christoffel symbols on the Greek ones. For example,

$$
\begin{aligned}
\nabla_\nu T^{\rho\mu}{}_i &= \partial_\nu T^{\rho\mu}{}_i + \widetilde{\Gamma}_{\nu\sigma}{}^\rho T^{\sigma\mu}{}_i + \widetilde{\Gamma}_{\nu\sigma}{}^\mu T^{\rho\sigma}{}_i - \omega_{\nu i}{}^j T^{\rho\mu}{}_j, \\
\nabla_\nu R^{\rho\mu}{}_{ij} &= \partial_\nu R^{\rho\mu}{}_{ij} + \widetilde{\Gamma}_{\nu\sigma}{}^\rho R^{\sigma\mu}{}_{ij} + \widetilde{\Gamma}_{\nu\sigma}{}^\mu R^{\rho\sigma}{}_{ij} - \omega_{\nu i}{}^k R^{\rho\mu}{}_{kj} - \omega_{\nu j}{}^k R^{\rho\mu}{}_{ik}.
\end{aligned}
$$

The first condition on the class of solutions of the equilibrium equations is that they permit solutions describing the presence of only dislocations in media. This means the

existence of solutions with zero curvature tensor corresponding to the absence of disclinations. Substitution of the condition $R_{\mu\nu}{}^{ij}=0$ into Eqn (20) for the $SO(3)$ connection yields

$$\begin{aligned} (12\kappa+2\beta_1-\beta_2-2\beta_3)T_i&=0,\\ (\kappa-\beta_1-\beta_2)T^*&=0,\\ (4\kappa+2\beta_1-\beta_2)W_{ijk}&=0. \end{aligned} \tag{21}$$

Here T_i, T^*, and W_{ijk} are the irreducible components of the torsion tensor,

$$T_{ijk}=W_{ijk}+T^*\epsilon_{ijk}+\frac{1}{2}(\delta_{ik}T_j-\delta_{jk}T_i),$$

where

$$\begin{aligned} T^*&=\frac{1}{6}T_{ijk}\epsilon^{ijk}, & T_j&=T_{ij}{}^i,\\ W_{ijk}&=T_{ijk}-T^*\epsilon_{ijk}-\frac{1}{2}(\delta_{ik}T_j-\delta_{jk}T_i), & W_{ijk}\epsilon^{ijk}&=W_{ij}{}^i=0. \end{aligned}$$

In a general case of dislocations all irreducible components of torsion tensor differ from zero (T_i, T^*, $W_{ijk}\neq 0$) and Eqns (21) have a unique solution

$$\beta_1=-\kappa,\qquad \beta_2=2\kappa,\qquad \beta_3=4\kappa. \tag{22}$$

For these coupling constants, the first four terms in Lagrangian (18) are equal to the Hilbert–Einstein Lagrangian $\kappa\widetilde{R}(e)$ up to a total divergence due to identity (130). Equation (19) then reduces to the Einstein equations with a cosmological constant

$$\widetilde{R}_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\widetilde{R}-\frac{\Lambda}{2\kappa}g_{\mu\nu}=0. \tag{23}$$

In this way, the first condition is satisfied.

According to the second condition, the equations of equilibrium must allow solutions with zero torsion $T_{\mu\nu}{}^i=0$. In this case, the curvature tensor has additional symmetry $R_{ijkl}=R_{klij}$, and Eqn (20) becomes

$$\begin{aligned} (\gamma_1+\gamma_2+\frac{1}{4}\gamma_3)\nabla_\nu\left(R^{S\nu}{}_ie^\mu{}_j-R^{S\mu}{}_ie^\nu{}_j-R^{S\nu}{}_je^\mu{}_i+R^{S\mu}{}_je^\nu{}_i\right)&\\ +\frac{1}{6}(\gamma_1+\gamma_2+4\gamma_3)\left(e^\nu{}_ie^\mu{}_j-e^\mu{}_ie^\nu{}_j\right)\nabla_\nu R=0.& \end{aligned} \tag{24}$$

Here, we decompose the Ricci tensor onto irreducible components,

$$R_{ij}=R^S{}_{ij}+R^A{}_{ij}+\frac{1}{3}R\delta_{ij},$$

where

$$R^S{}_{ij}=R^S{}_{ji},\qquad R^{Si}{}_i=0,\qquad R^A{}_{ij}=-R^A{}_{ji}.$$

Note that for zero torsion, the Ricci tensor is symmetrical: $R^A{}_{ij}=0$. Contraction of Eqn (24) with $e_\mu{}^j$ leads to the equation

$$(\gamma_1+\gamma_2+\frac{1}{4}\gamma_3)\nabla_\nu R^{S\nu}{}_\mu+\frac{1}{3}(\gamma_1+\gamma_2+4\gamma_3)\nabla_\mu R=0.$$

In the general case of nonvanishing curvature, the covariant derivatives $\nabla_\nu R^{S\nu}{}_\mu$ and $\nabla_\mu R$ differ from zero and are independent. Therefore, we obtain two equations for the coupling constants,

$$\gamma_1 + \gamma_2 + \frac{1}{4}\gamma_3 = 0, \qquad \gamma_1 + \gamma_2 + 4\gamma_3 = 0.$$

which have a unique solution

$$\gamma_1 = -\gamma_2 = \gamma, \qquad \gamma_3 = 0. \tag{25}$$

In this case, Eqn (19) for the vielbein corresponding to a nonzero torsion also reduces to Einstein equations (23).

The last requirement for the existence of solutions with zero curvature and torsion is satisfied only for the zero cosmological constant $\Lambda = 0$.

Therefore, the simple and physically reasonable requirements define the two parameter Lagrangian [22]

$$\frac{1}{e}L = -\kappa\widetilde{R} + 2\gamma R^A{}_{ij}R^{Aij}, \tag{26}$$

which is the sum of the Hilbert–Einstein Lagrangian for the vielbein and the square of the antisymmetric part of the Ricci tensor. We note that $\widetilde{R}(e)$ and $R^A{}_{ij}(e,\omega)$ are constructed for different metrical connections.

Other quadratic Lagrangians were considered, for example, in [9, 63]. We note that they also contain the displacement vector as an independent variable along with the vielbein.

Expression (26) defines the free energy density in the geometric theory of defects and leads to the equilibrium equations (the Euler–Lagrange equations). In our geometric approach, the displacement vector and spin structure do not enter expression (26) for the free energy.

6 Gauge fixing

In the geometric approach, the vielbein $e_\mu{}^i$ and $\mathbb{SO}(3)$ connection $\omega_{\mu i}{}^j$ are the only variables. The displacement field u^i and the spin structure $\omega_i{}^j$ can be introduced only in those regions of media where defects are absent. As the consequence of the absence of disclinations $R_{\mu\nu i}{}^j = 0$, the $\mathbb{SO}(3)$ connection is actually a pure gauge (127), i.e., the spin structure $\omega_i{}^j$ exists. If, in addition, dislocations are absent ($T_{\mu\nu}{}^i = 0$) then there is the displacement field such that the vielbein equals its partial derivatives (128). In this and only in this case can we introduce the displacement field and spin structure. We show below that this can be done such that the equations of nonlinear elasticity theory and the principal chiral $\mathbb{SO}(3)$ field are fulfilled.

For free energy in (26), the Euler–Lagrange equations are covariant under general coordinate transformations in $\mathbb{R}^3$ and local $\mathbb{SO}(3)$ rotations. This means that any solution of the equilibrium equations is defined up to diffeomorphisms and local rotations. For the geometric theory of defects to make predictions, we have to fix the coordinate system (to fix the gauge in the language of gauge field theory). This allows us to choose a unique representative from each class of equivalent solutions. We say afterwards that this solution

of the Euler–Lagrange equations describes the distribution of defects in the laboratory coordinate system.

We start with gauge-fixing the diffeomorphisms. For this, we choose the elastic gauge proposed in [23]. This question is of primary importance, and we discuss it in detail.

Equations (2), (3) of elasticity theory for $f^i = 0$ yield the second-order equation for the displacement vector

$$(1-2\sigma)\triangle u_i + \partial_i\partial_j u^j = 0, \tag{27}$$

where

$$\sigma = \frac{\lambda}{2(\lambda+\mu)}$$

is the Poisson ratio, $(-1 \le \sigma \le 1/2)$, and $\triangle$ is the Laplace operator. It can be rewritten in terms of the induced metric (5), for which we obtain a first-order equation. We choose precisely this equation as the gauge condition fixing the diffeomorphisms. We note that the gauge condition is not uniquely defined because the induced metric is nonlinear in the displacement vector, and different equations for the metric may have the same linear approximation. We give two possible choices,

$$g^{\mu\nu}\overset{\circ}{\nabla}_\mu g_{\nu\rho} + \frac{\sigma}{1-2\sigma} g^{\mu\nu}\overset{\circ}{\nabla}_\rho g_{\mu\nu} = 0, \tag{28}$$

$$\overset{\circ}{g}{}^{\mu\nu}\overset{\circ}{\nabla}_\mu g_{\nu\rho} + \frac{\sigma}{1-2\sigma} \overset{\circ}{\nabla}_\rho g^T = 0, \tag{29}$$

where we introduced the notation for the trace of metric $g^T = \overset{\circ}{g}{}^{\mu\nu} g_{\mu\nu}$. Gauge conditions (28) and (29) are understood in the following way. The metric $\overset{\circ}{g}_{\mu\nu}$ is the Euclidean metric written in an arbitrary coordinate system, for example, in the cylindrical or spherical coordinate system. The covariant derivative $\overset{\circ}{\nabla}_\mu$ is built from the Christoffel symbols corresponding to the metric $\overset{\circ}{g}_{\mu\nu}$, and $\overset{\circ}{\nabla}_\mu \overset{\circ}{g}_{\nu\rho} = 0$ as a consequence. The metric $g_{\mu\nu}$ is the metric describing dislocations [an exact solution of the equilibrium equations for the free energy (26)]. The gauge conditions differ because in the first and second cases the contraction is performed with the metric of dislocation $g^{\mu\nu}$ and the Euclidean metric $\overset{\circ}{g}{}^{\mu\nu}$, respectively, without changing the linear approximation. Both gauge conditions yield Eqn (27) in the linear approximation in the displacement vector (5). This is most easily verified in Cartesian coordinates.

From the geometric standpoint, we have the following. The medium with dislocations is diffeomorphic to the Euclidean space $\mathbb{R}^3$ equipped with two metrics $\overset{\circ}{g}_{\mu\nu}$ and $g_{\mu\nu}$. The metric $\overset{\circ}{g}_{\mu\nu}$ is a flat Euclidean metric written in an arbitrary coordinate system. The metric $g_{\mu\nu}$ is not flat and describes the distribution of dislocations in the same coordinate system. In fact, the metric $\overset{\circ}{g}_{\mu\nu}$ is used only to fix the coordinate system in which the metric $g_{\mu\nu}$ is measured.

If the solution of the equilibrium equations satisfies one of the gauge conditions (28), (29), written, for example, in cylindrical coordinate system then we say that the solution is found in the cylindrical coordinates. We suppose here that the distribution of dislocations in elastic media in the laboratory cylindrical coordinate system is described by this particular solution. Analogously, we may seek solutions in a Cartesian, spherical, or any other coordinate system.

Gauge conditions may also be written for the vielbein $e_\mu{}^i$, which is defined by Eqn (120). This involves additional arbitrariness because the vielbein is defined up to local rotations. This invariance leads to different linear approximations for the vielbein in terms of the displacement vector. We consider two possibilities in Cartesian coordinates:

$$e_{\mu i} \approx \delta_{\mu i} - \partial_\mu u_i, \tag{30}$$

$$e_{\mu i} \approx \delta_{\mu i} - \frac{1}{2}(\partial_\mu u_i + \partial_i u_\mu), \tag{31}$$

where the index is lowered with the help of the Kronecker symbol. For these possibilities and gauge condition (29) we have two gauge conditions for the vielbein

$$\overset{\circ}{g}{}^{\mu\nu}\overset{\circ}{\nabla}_\mu e_{\nu i} + \frac{1}{1-2\sigma}\overset{\circ}{e}{}^\mu{}_i \overset{\circ}{\nabla}_\mu e^T = 0, \tag{32}$$

$$\overset{\circ}{g}{}^{\mu\nu}\overset{\circ}{\nabla}_\mu e_{\nu i} + \frac{\sigma}{1-2\sigma}\overset{\circ}{e}{}^\mu{}_i \overset{\circ}{\nabla}_\mu e^T = 0, \tag{33}$$

where $e^T = \overset{\circ}{e}{}^\mu{}_i e_\mu{}^i$. These conditions differ in the coefficient before the second term. We note that in a curvilinear coordinate system, the covariant derivative $\overset{\circ}{\nabla}_\mu$ must also include the flat $\mathbb{SO}(3)$ connection acting on indices i, j. One can also write other possible gauge conditions having the same linear approximation. The question of the correct choice is unanswered at present and outside the scope of this review. At the moment, we want only to demonstrate that the system of coordinates must be fixed, and that the gauge condition depends on the Poisson ratio, which is the experimentally observed quantity.

Gauge conditions (32)–(33) are first order equations by themselves and have some arbitrariness. Therefore, to fix a solution uniquely, we must impose additional boundary conditions on the vielbein for any given problem.

If the defect-free case, $T_{\mu\nu}{}^i = 0$, $R_{\mu\nu j}{}^i = 0$, and the equilibrium equations are satisfied because the Euler–Lagrange equations for (26) are satisfied. In this and only in this case, we can introduce a displacement vector, and the elastic gauge reduces to the equations of nonlinear elasticity theory. In the presence of defects, a displacement field does not exist, and the elastic gauge simply defines the vielbein.

In choosing the free energy functional, we required that the conditions $R_{\mu\nu}{}^{ij} = 0$ and $T_{\mu\nu}{}^i = 0$ satisfy the Euler–Lagrange equations. This is important because otherwise we would obtain an additional condition on the displacement vector (the Euler–Lagrange equations) besides the elasticity theory equations following from the elastic gauge.

We stress an important point once again. In the geometric theory of defects, we assume that there is a preferred laboratory coordinate system in which measurements are made. This coordinate system is related to the medium without defects and elastic stresses and corresponds to the flat Euclidean space $\mathbb{R}^3$. Gauge conditions (28), (29) and (32), (33) are written precisely in this Euclidean space $\mathbb{R}^3$ and contain a measurable quantity, the Poisson ratio σ. This property essentially distinguishes the geometric theory of defects from the models of gravity in which all coordinate systems are considered equivalent.

The elastic gauge is used to fix diffeomorphisms. The expression for the free energy in (26) is also invariant under local $\mathbb{SO}(3)$ rotations, and they must also be fixed. For this, we recently proposed the Lorentz gauge for the connection [24]

$$\partial_\mu \omega_{\mu j}{}^i = 0. \tag{34}$$

This gauge is written in the laboratory Cartesian coordinate system and has deep physical meaning. That is, let disclinations be absent ($R_{\mu\nu j}{}^i = 0$). Then the $\mathbb{SO}(3)$ connection is a pure gauge:

$$\omega_{\mu j}{}^i = \partial_\mu S^{-1}{}_j{}^k S_k{}^i, \qquad S_j{}^i \in \mathbb{SO}(3).$$

In this case, the Lorentz gauge reduces to the principal chiral $\mathbb{SO}(3)$-field equations

$$\partial_\mu(\partial_\mu S^{-1}{}_j{}^k S_k{}^i) = 0$$

for the spin structure $\omega^{ij}(x)$. Principal chiral field models (see, e.g., [64–68]) for different groups and in a different number of dimensions attract much interest in mathematical physics because they admit solutions of topological soliton types and find broad applications in physics.

Thus, the Lorentz gauge (34) means the following. In the absence of disclinations, equations of equilibrium are identically satisfied, and there exists a field ω^{ij} that satisfies equations for the principal chiral field. By this we mean that the spin structure of the medium is described by the model of the principal chiral field in the defect-free case.

The principal chiral field model is not the only one that can be used for fixing local rotations. The Skyrme model [69] can also be used for this purpose. The Euler–Lagrange equations for this model are not difficult to rewrite in terms of the $\mathbb{SO}(3)$ connection and use as the gauge conditions.

There are other models for spin structures. For describing the distribution of magnetic moments in ferromagnets or the director field in liquid crystals, one uses the expression for the free energy depending on the vector n-field itself [70, 61]. Lately, much attention is paid to the Faddeev model of the n-field [71]. The question whether there are gauge conditions on the $\mathbb{SO}(3)$ connection that yield these models in the absence of disclinations is unanswered at present.

Thus, we pose the following problem in the geometric theory of defects: to find the solution of the Euler–Lagrange equations for free energy (26) that satisfies the elastic gauge for the vielbein and the Lorentz gauge for the $\mathbb{SO}(3)$ connection. In sections 8 and 11, we solve this problem for the wedge dislocation in the framework of the classical elasticity theory and the geometric theory of defects, respectively, and afterwards compare the obtained results.

7 Asymmetric elasticity theory

In the preceeding section, we used the elasticity theory and the principal chiral $\mathbb{SO}(3)$-field model to fix the invariance of free energy (26) in the geometric theory of defects. This is not a unique possibility, because other models may be used for gauge fixing. In the present section, we show how another model – asymmetric elasticity theory – can be used for fixing diffeomorphisms and local rotations.

At the beginning of the last century, the Cosserat brothers developed the theory of elastic media, every point of which is characterized not only be its position but also by its orientation in space [72], i.e., a vielbein is specified at every point (Fig.8).

From physical standpoint, this means that every atom in the crystalline structure is not a point but an extended object having orientation. In this case, the stress tensor is no longer a symmetric tensor, and the corresponding theory is called the asymmetric

elasticity theory. Contemporary exposition of this approach is given in [73]. In the present section, we show that the asymmetric theory of elasticity is naturally incorporated into the geometric theory of defects.

The main variables in the asymmetric theory of elasticity are the displacement vector $u^i(x)$ and the rotation angle $\omega^i(x)$. The direction of the pseudovector ω^i coincides with the rotation axis of the medium element and its length is equal to the angle of rotation. The angle of rotation discussed in section 4 is dual to the spin structure field $\omega^{ij}(x)$: $\omega_{ij} = \varepsilon_{ijk}\omega^k$.

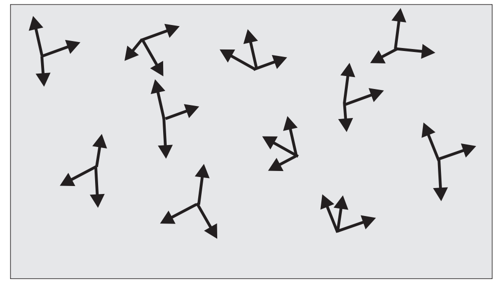

Figure 8: Every point of the Cosserat medium is characterized not only by its position but also by its orientation in space.

The Cosserat medium is characterized by the stress tensor $\sigma^{ij}(x)$ (the density of forces acting on the surface with normal i in the direction j) and the torque stress tensor $\mu^{ij}(x)$ (the density of torques acting on the surface with normal i in the direction j). The Cosserat medium is in equilibrium if forces and torques are balanced at every point,

$$\partial_j \sigma^{ji} + f^i = 0, \tag{35}$$
$$\varepsilon^{ijk}\sigma_{jk} + \partial_j \mu^{ji} + m^i = 0, \tag{36}$$

where $f^i(x)$ and $m^i(x)$ are the densities of nonelastic external forces and torques. As a consequence of Eqn (36), the stress tensor is symmetric if and only if the condition $\partial_j \mu^{ji} + m^i = 0$ is satisfied.

The displacement field and rotation angle uniquely define the deformation tensor $\epsilon_{ij}(x)$ and the twist tensor $\kappa_{ij}(x)$:

$$\begin{aligned} \epsilon_{ij} &= \partial_i u_j - \omega_{ij}, \\ \kappa_{ij} &= \partial_i \omega_j. \end{aligned} \tag{37}$$

In general, the deformation and twist tensors have no symmetry in their indices.

Hook's law in the Cosserat medium is changed to two linear relations connecting the stress and torque tensors with the deformation and twist tensors,

$$\sigma_{ij} = 2\mu\epsilon_{\{ij\}} + 2\alpha\epsilon_{[ij]} + \lambda\delta_{ij}\epsilon_k{}^k, \tag{38}$$
$$\mu_{ij} = 2\gamma\kappa_{\{ij\}} + 2\epsilon\kappa_{[ij]} + \beta\delta_{ij}\kappa_k{}^k, \tag{39}$$

where μ, λ are the Lamé coefficients, and $\alpha, \beta, \gamma, \epsilon$ are four new elastic constants characterizing the medium. Braces and square brackets denote symmetrization and antisymmetrization of indices, respectively.

In the case where

$$\omega_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i), \tag{40}$$

the deformation tensor is symmetric and has the previous form (4). Equation (36), together with (38) and (39), then reduces to the equation

$$(\gamma + \epsilon)\varepsilon^{ijk}\triangle\partial_j u_k + m^i = 0.$$

The first term vanishes as a consequence of Eqn (27). Thus, for spin structure (40) and $m^i = 0$ we return to the symmetric elasticity theory.

Equations (35), (36), (38), and (39), together with the boundary conditions, define the equilibrium state of Cosserat media. We now show how this model is included in the geometric theory. First, we note that in the absence of defects ($T_{\mu\nu}{}^i = 0$, $R_{\mu\nu}{}^{ij} = 0$), the fields u^i and ω^{ij} exist. Then the vielbein and the $\mathbb{SO}(3)$ connection are defined by the deformation and twist tensors in the linear approximation:

$$e_\mu{}^i = \partial_\mu y^j S_j{}^i(\omega) \approx (\delta_\mu^j - \partial_\mu u^j)(\delta_j^i + \omega_j{}^i) \approx \delta_\mu^i - \epsilon_\mu{}^i, \tag{41}$$

$$\omega_\mu{}^{ij} \approx \partial_\mu \omega^{ij} = \varepsilon^{ijk}\kappa_{\mu k}. \tag{42}$$

We note that relations (37) can be regarded as equations for the displacement vector and rotation angle. The corresponding integrability conditions were obtained in [74]. These integrability conditions are the linear approximations of equalities $T_{\mu\nu}{}^i = 0$ and $R_{\mu\nu}{}^{ij} = 0$ defining the absence of defects.

If nonelastic forces and torques are absent ($f^i = 0$, $m^i = 0$), then the asymmetric theory of elasticity reduces to second-order equations for the displacement vector and rotation angle:

$$(\mu + \alpha)\triangle u^i + (\mu - \alpha + \lambda)\partial^i\partial_j u^j - 2\alpha\partial_j\omega^{ji} = 0, \tag{43}$$

$$(\gamma + \epsilon)\triangle\omega^i + (\gamma - \epsilon + \beta)\partial^i\partial_j\omega^j + 2\alpha\varepsilon^{ijk}(\partial_j u_k - \omega_{jk}) = 0. \tag{44}$$

We rewrite these equations for the vielbein and $\mathbb{SO}(3)$ connection

$$(\mu + \alpha)\overset{\circ}{\nabla}{}^\mu e_\mu{}^i + (\mu - \alpha + \lambda)\overset{\circ}{\nabla}{}^i e^T - (\mu - \alpha)\omega_\mu{}^{\mu i} = 0, \tag{45}$$

$$\frac{1}{2}(\gamma + \epsilon)\varepsilon^{ijk}\overset{\circ}{\nabla}{}^\mu\omega_{\mu jk} + \frac{1}{2}(\gamma - \epsilon + \beta)\varepsilon^{\mu jk}\overset{\circ}{\nabla}{}^i\omega_{\mu jk} + 2\alpha\varepsilon^{i\mu j}e_{\mu j} = 0. \tag{46}$$

Of course, these are not the only equations that coincide with Eqns (43) and (44) in the linear approximation. At present, we do not have arguments for the unique choice. The derived nonlinear equations of the asymmetric elasticity theory can be used as gauge conditions in the geometric theory of defects. Here, we have six equations for fixing diffeomorphisms (three parameters) and local $\mathbb{SO}(3)$ rotations (three parameters). Thus, the asymmetric theory of elasticity is naturally embedded in the geometric theory of defects.

In section 6, we considered the elastic gauge for the vielbein and the Lorentz gauge for the $\mathbb{SO}(3)$ connection. In this case, the spin structure variables do not interact with elastic deformations when defects are absent. In the asymmetric elasticity theory, the elastic stresses directly influence the spin structure and vice versa.

8 Wedge dislocation in the elasticity theory

By wedge dislocation, we understand an elastic medium that is topologically the Euclidean space $\mathbb{R}^3$ without the $z = x^3$ axis – the core of dislocation, obtained in the following way. We consider the infinite elastic medium without defects and cut out the infinite wedge of angle $-2\pi\theta$. For definiteness, we assume that the edge of the wedge

coincides with the z axis (Fig.9). The edges of the cut are then moved symmetrically one to the other and glued together. After that, the medium moves to the equilibrium state under the action of elastic forces. If the wedge is cut out from the medium, then the angle is considered negative: $-1 < \theta < 0$. For positive θ, the wedge is added. Thus, the elastic medium initially occupies a domain greater or lesser than the Euclidean space $\mathbb{R}^3$, depending on the sign of the deficit angle θ; in the cylindrical coordinates r, φ, z this domain is described by the inequalities

$$0 \le r < \infty, \quad 0 \le \varphi \le 2\pi\alpha, \quad -\infty < z < \infty, \qquad \alpha = 1 + \theta. \tag{47}$$

We note that the wedge dislocation is often called disclination. In our approach, this term seems unnatural because the wedge dislocation is related to a nontrivial torsion. Moreover, the term disclination is used for defects in the spin structure.

We now proceed with the mathematical formulation of the problem for the wedge dislocation in the framework of elasticity theory. To avoid divergent expressions arising for infinite media, we suppose that the wedge dislocation is represented by the cylinder of a finite radius R. This problem has translational symmetry along the z axis and rotational symmetry in the x, y plane. Therefore, we use cylindrical coordinate system. Let

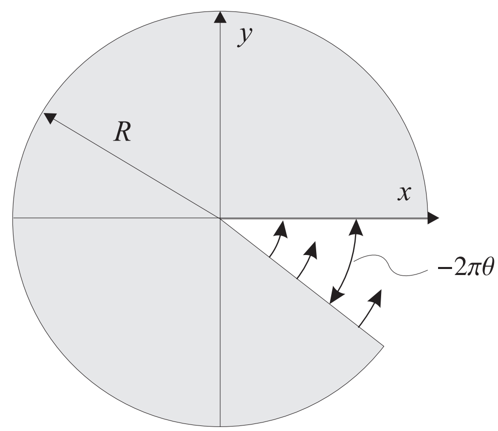

Figure 9: Wedge dislocation with the deficit angle $2\pi\theta$. For negative and positive θ, the wedge is cut out or added, respectively.

$$\hat{u}_i = (\hat{u}_r, \hat{u}_\varphi, \hat{u}_z) \tag{48}$$

be the components of the displacement covector with respect to orthonormal basis in the cylindrical coordinate system. This covector satisfies the equilibrium equation following the substitution of (3) in Eqn (2) in domain (47),

$$(1 - 2\sigma)\triangle\hat{u}_i + \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_j \hat{u}^j = 0, \tag{49}$$

where $\overset{\circ}{\nabla}_i$ is the covariant derivative for the flat Euclidean metric in the considered coordinate system.

For references, we write expressions for the divergence and Laplacian for a covector field in the cylindrical coordinate system:

$$\begin{aligned}
\overset{\circ}{\nabla}_i \hat{u}^i &= \frac{1}{r}\partial_r(r\hat{u}^r) + \frac{1}{r}\partial_\varphi \hat{u}^\varphi + \partial_z \hat{u}^z, \\
\triangle\hat{u}_r &= \frac{1}{r}\partial_r(r\partial_r\hat{u}_r) + \frac{1}{r^2}\partial_\varphi^2\hat{u}_r + \partial_z^2\hat{u}_r - \frac{1}{r^2}\hat{u}_r - \frac{2}{r^2}\partial_\varphi\hat{u}_\varphi, \\
\triangle\hat{u}_\varphi &= \frac{1}{r}\partial_r(r\partial_r\hat{u}_\varphi) + \frac{1}{r^2}\partial_\varphi^2\hat{u}_\varphi + \partial_z^2\hat{u}_\varphi - \frac{1}{r^2}\hat{u}_\varphi + \frac{2}{r^2}\partial_\varphi\hat{u}_r, \\
\triangle\hat{u}_z &= \frac{1}{r}\partial_r(r\partial_r\hat{u}_z) + \frac{1}{r^2}\partial_\varphi^2\hat{u}_z + \partial_z^2\hat{u}_z.
\end{aligned}$$

Taking the symmetry of the problem into account, we seek the solution of Eqn (49) in the form

$$\hat{u}_r = u(r), \qquad \hat{u}_\varphi = A(r)\varphi, \qquad \hat{u}_z = 0$$

where $u(r)$ and $A(r)$ are two unknown functions depending only on the radius. We impose the boundary conditions:

$$\hat{u}_r|_{r=0} = 0, \qquad \hat{u}_\varphi|_{r=0} = 0, \qquad \hat{u}_\varphi|_{\varphi=0} = 0, \qquad \hat{u}_\varphi|_{\varphi=2\pi\alpha} = -2\pi\theta r, \qquad \partial_r \hat{u}_r|_{r=R} = 0. \quad (50)$$

The first four equations are geometrical and correspond to the process of dislocation creation. The last condition has simple physical meaning: the absence of external forces at the boundary of the medium. The unknown function $A(r)$ is found from the second to the last boundary condition (50)

$$A(r) = -\frac{\theta}{1+\theta} r.$$

Straightforward substitution then shows that φ and z components of equation of equilibrium (49) are identically satisfied, and the radial component reduces to the equation

$$\partial_r(r\partial_r u) - \frac{u}{r} = D, \qquad D = -\frac{1-2\sigma}{1-\sigma}\frac{\theta}{1+\theta} = \mathsf{const}.$$

The general solution of this equation is

$$u = \frac{D}{2} r \ln r + c_1 r + \frac{c_2}{r}, \qquad c_{1,2} = \mathsf{const}.$$

The constant of integration $c_2 = 0$ due to the boundary condition at zero. The constant c_1 is found from the last boundary condition in (50). Finally, we obtain the known solution of the considered problem [62]

$$\begin{aligned} \hat{u}_r &= \frac{D}{2} r \ln \frac{r}{\mathsf{e}R}, \\ \hat{u}_\varphi &= -\frac{\theta}{1+\theta} r\varphi. \end{aligned} \qquad (51)$$

The letter e in the expression for $\hat{u}_r$ denotes the base of the natural logarithm. We note that the radial component of the displacement vector diverges as $R \to \infty$. This means that the description of the wedge dislocation requires considering a finite-radius cylinder.

The linear elasticity theory is applicable for small relative displacements, which for a wedge dislocation are equal to

$$\frac{d\hat{u}_r}{dr} = -\frac{\theta}{1+\theta}\frac{1-2\sigma}{2(1-\sigma)} \ln \frac{r}{R}, \qquad \frac{1}{r}\frac{d\hat{u}_\varphi}{d\varphi} = -\frac{\theta}{1+\theta}.$$

This means that we are able to expect correct results for the displacement field for small deficit angles ($\theta \ll 1$) and near the boundary of the cylinder ($r \sim R$).

We find the metric induced by the wedge dislocation in the linear approximation in the deficit angle θ. Calculations can be performed using the general formulas (5) or the known expression for the variation of the form of the metric

$$\delta g_{\mu\nu} = -\overset{\circ}{\nabla}_\mu u_\nu - \overset{\circ}{\nabla}_\nu u_\mu. \qquad (52)$$

After simple calculations, we obtain the expression for the metric in the x, y plane:

$$ds^2 = \left(1 + \theta\frac{1-2\sigma}{1-\sigma}\ln\frac{r}{R}\right)dr^2 + r^2\left(1 + \theta\frac{1-2\sigma}{1-\sigma}\ln\frac{r}{R} + \theta\frac{1}{1-\sigma}\right)d\varphi^2. \qquad (53)$$

This metric is compared with the metric obtained as the solution of three-dimensional Einstein equations in section 11.

9 Edge dislocation in the elasticity theory

Wedge dislocations are relatively rarely met in nature because they require a much amount of a medium to be added or removed, resulting in a vast quantity of energy expenses. Nevertheless, their study is of great importance because other linear dislocations can be expressed as a superposition of wedge dislocations. In this respect, wedge dislocations are elementary. We show this for an edge dislocation – one of the most widely spread dislocations, as an example. An edge dislocation with the core coinciding with the z axis is shown in Fig.10a. It appears as the result of cutting the medium over the half-plane

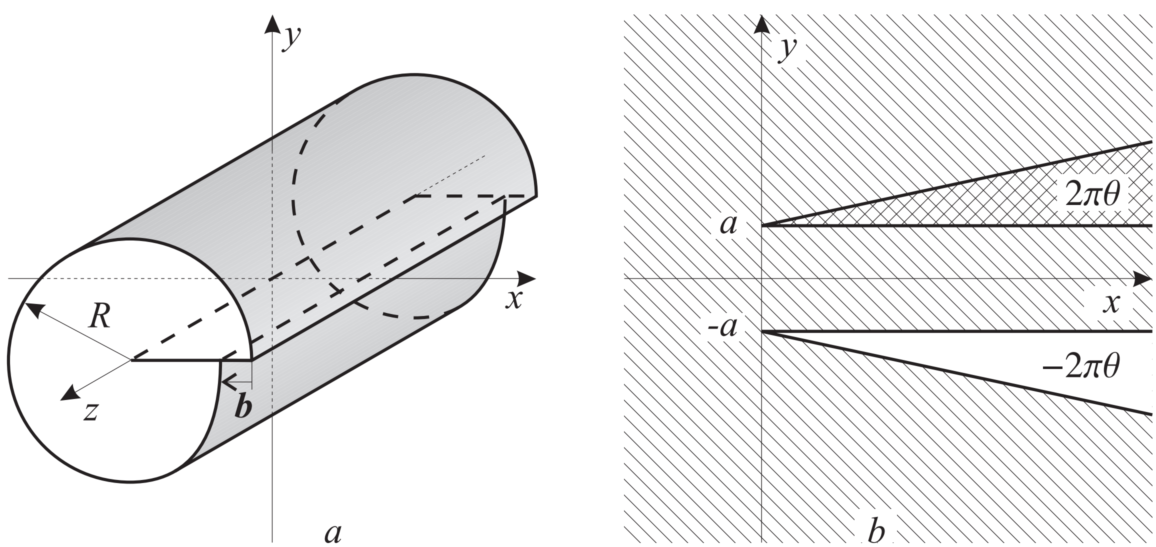

Figure 10: Edge dislocation with the Burgers vector $\boldsymbol{b}$ directed to the dislocation axis (a). Edge dislocation as the dipole of two wedge dislocations with positive and negative deficit angles (b).

$y = 0$, $x > 0$, moving the lower edge of the cut towards the z axis on a constant (far from the core of the dislocation) Burgers vector $\boldsymbol{b}$, and subsequently gluing the edges. To find the displacement vector field for the edge dislocation, we may solve the corresponding boundary value problem for the equilibrium equations (49) [61]. However, we follow another way, knowing the explicit form of the displacement vector for a wedge dislocation. The edge dislocation is represented by the dipole of two wedge dislocations with positive, $2\pi\theta$, and negative, $-2\pi\theta$, deficit angles as shown in Fig.10b. We assume that the axes of the first and second wedge dislocations are parallel to the z axis and intersect the x, y plane at points with the respective coordinates $(0, a)$ and $(0, -a)$. The distance between the wedge dislocation axes is equal to $2a$. It follows from the expression for the displacement

field (51) that far away from the origin ($r \gg a$), the displacement field for the wedge dislocations has the following form in the first order in small θ and a/r:

$$\begin{aligned} u_x^{(1)} &\approx -\theta\left[\frac{1-2\sigma}{2(1-\sigma)}x\ln\frac{r-a\sin\varphi}{\mathrm{e}R}-(y-a)\left(\varphi-\frac{a\cos\varphi}{r}\right)\right], \\ u_y^{(1)} &\approx -\theta\left[\frac{1-2\sigma}{2(1-\sigma)}(y-a)\ln\frac{r-a\sin\varphi}{\mathrm{e}R}+x\left(\varphi-\frac{a\cos\varphi}{r}\right)\right], \end{aligned} \tag{54}$$

$$\begin{aligned} u_x^{(2)} &\approx \quad\theta\left[\frac{1-2\sigma}{2(1-\sigma)}x\ln\frac{r+a\sin\varphi}{\mathrm{e}R}-(y+a)\left(\varphi+\frac{a\cos\varphi}{r}\right)\right], \\ u_y^{(2)} &\approx \quad\theta\left[\frac{1-2\sigma}{2(1-\sigma)}(y+a)\ln\frac{r+a\sin\varphi}{\mathrm{e}R}+x\left(\varphi+\frac{a\cos\varphi}{r}\right)\right]. \end{aligned} \tag{55}$$

It is sufficient to sum displacement fields (54) and (55) to find the displacement field for the edge dislocation because the elasticity theory equations are linear. After simple calculations, up to the translation of the whole medium by a constant vector along the y axis, we obtain

$$\begin{aligned} u_x &= \quad b\left[\operatorname{arctg}\frac{y}{x}+\frac{1}{2(1-\sigma)}\frac{xy}{x^2+y^2}\right], \\ u_y &= -b\left[\frac{1-2\sigma}{2(1-\sigma)}\ln\frac{r}{\mathrm{e}R}+\frac{1}{2(1-\sigma)}\frac{x^2}{x^2+y^2}\right], \end{aligned} \tag{56}$$

where we have introduced the notation for the modulus of the Burgers vector

$$b = |\boldsymbol{b}| = -2a\theta.$$

This result coincides with the expression for the displacement field obtained by direct solution of the elasticity theory equations [61]. Thus, we have shown that an edge dislocation is the dipole of two parallel wedge dislocations.

We next find the metric induced by the edge dislocation. Using formulas (52), we obtain the metric in the x, y plane in the linear approximation in θ and a/r:

$$dl^2 = \left(1+\frac{1-2\sigma}{1-\sigma}\frac{b}{r}\sin\varphi\right)\left(dr^2+r^2d\varphi^2\right)-\frac{2b\cos\varphi}{1-\sigma}drd\varphi. \tag{57}$$

We note that the induced metric for an edge dislocation does not depend on R.

10 Parallel wedge dislocations

In the absence of disclinations ($R_{\mu\nu i}{}^j = 0$), the $\mathbb{SO}(3)$ connection is a pure gauge, and equations of equilibrium for the $\mathbb{SO}(3)$ connection (20) are identically satisfied. In this case, the explicit form of the $\mathbb{SO}(3)$ connection is uniquely determined by the spin structure ω^{ij}. The field ω^{ij} satisfies equations for the principal chiral field as a consequence of the Lorentz gauge in (34). Solution of this system of equations defines the trivial $\mathbb{SO}(3)$ connection. Thus in the absence of disclinations, the problem is reduced to solution of the Einstein equations for the vielbein in the elastic gauge and solution of the principal

chiral field model for the spin structure. After that, we can compute the torsion tensor through the formulas (125), which defines the surface density of the Burgers vector.

Because disclinations are absent (the curvature tensor is equal to zero), we have the space of absolute parallelism. The whole geometry is then defined by the vielbein $e_\mu{}^i$, which uniquely defines the torsion tensor via (125) for a vanishing $\mathbb{SO}(3)$ connection. Here, we assume that a trivial $\mathbb{SO}(3)$ connection is equal to zero. The vielbein $e_\mu{}^i$ satisfies the three-dimensional Einstein equations with a Euclidean signature metric, which follow from the expression for the free energy (26) for $R_{\mu\nu j}{}^i = 0$,

$$\widetilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \widetilde{R} = T_{\mu\nu}. \tag{58}$$

Here, we have added the source of dislocations $T_{\mu\nu}$ to the right hand side of Einstein equations (it is the energy-momentum tensor in gravity).

We note that without a source of dislocations, the model would be trivial. Indeed, the scalar curvature and Ricci tensor are equal to zero: $\widetilde{R} = 0$, $\widetilde{R}_{\mu\nu} = 0$ for $T_{\mu\nu} = 0$ as a consequence of Einstein equations (58). Then, the full curvature tensor without sources is also equal to zero because in three-dimensional space, it is in a one-to-one correspondence with Ricci tensor (129). The vanishing of the full curvature tensor means the triviality of the model because defects are absent in this case. The similar statement in three-dimensional gravity is well known. It is usually formulated as: "Three-dimensional gravity does not describe dynamical, i.e., propagating degrees of freedom".

For our purposes, we have to find the solution of Einstein equations (58) describing one wedge dislocation. The Einstein equations are a system of nonlinear second-order partial differential equations. Not too many exact solutions are known at present, even in a three-dimensional space. The remarkable exact solution describing an arbitrary static distribution of point particles is well known in three-dimensional gravity for the Lorentz signature metric $(+--)$ [75–77]. We find this solution for the Euclidean signature metric and show that it describes an arbitrary distribution of parallel wedge dislocations in the geometric theory of defects. Hence, we first consider the more general case of an arbitrary number of wedge dislocations and then analyze in detail one wedge dislocation which is of interest to us. We do this deliberately because the solution in a more general case does not involve essential complications. At the same time, an arbitrary distribution of wedge dislocations is much more interesting for applications. For example, the edge dislocation was shown in the preceeding section to be represented by a dipole of two wedge dislocations of different signs.

We consider elastic medium with arbitrary distributed but parallel wedge dislocations. We choose the coordinate system such that the $z = x^3$ axis is parallel to dislocations axes, and axes $\{x^\alpha\} = \{x, y\}$, $\alpha = 1, 2$ are perpendicular to the z axis. Then the metric has the block-diagonal form

$$ds^2 = dl^2 + N^2 dz^2, \tag{59}$$

where

$$dl^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

is a two-dimensional metric on the x, y plane. A two-dimensional metric $g_{\alpha\beta}(x, y)$ and a function $N(x, y)$ are independent of z due to translational symmetry along the z axis.

We can say this more simpler dropping the physical arguments. We consider the block-diagonal metric of form (59), which has translational invariance along the z axis.

Afterwards, we show that the corresponding solution of the Einstein equations indeed describes an arbitrary distribution of parallel wedge dislocations.

The curvature tensor for metric (59) has the components:

$$\begin{aligned}
&\widetilde{R}_{\alpha\beta\gamma}{}^{\delta}=R^{(2)}_{\alpha\beta\gamma}{}^{\delta}, && \widetilde{R}_{\alpha z\gamma}{}^{z}=\frac{1}{N}\nabla_\alpha\nabla_\gamma N,\\
&\widetilde{R}_{\alpha\beta\gamma}{}^{z}=\widetilde{R}_{\alpha z\gamma}{}^{\delta}=0,
\end{aligned}$$

where $R^{(2)}_{\alpha\beta\gamma}{}^{\delta}$ is the curvature tensor for the two-dimensional metric $g_{\alpha\beta}$ and ∇_α is the two-dimensional covariant derivative with the Christoffel symbols.

We choose the source of dislocations as

$$\begin{aligned}
T_{zz}&=\frac{2\pi}{\sqrt{g^{(2)}}}\sum_{n=1}^{M}\theta_n\delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}_n),\\
T_{\alpha\beta}&=T_{\alpha z}=T_{z\alpha}=0,
\end{aligned}\tag{60}$$

where $\delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}_n)=\delta(x-x_n)\delta(y-y_n)$ is the two-dimensional δ-function on the x,y plane with the support at a point $\boldsymbol{r}_n=(x_n,y_n)$. The factor $g^{(2)}=\mathsf{det}\,g_{\alpha\beta}$ in front of the sum sign is due to the property of the δ-function, which is not a function but a tensor density with respect to general coordinate transformations. We show later that the solution of the Einstein equations with such a source describes M parallel wedge dislocations with deficit angles θ_n, which intersect the x,y plane at the points (x_n,y_n). In three-dimensional gravity this source corresponds to particles of masses $m_n=2\pi\theta_n$, being at rest at the points $\boldsymbol{r}_n$.

Einstein equations (58) then reduce to four equations,

$$\nabla_\alpha\nabla_\beta N-g_{\alpha\beta}\nabla^\gamma\nabla_\gamma N=0,\tag{61}$$

$$-\frac{1}{2}N^3R^{(2)}=\frac{2\pi}{\sqrt{g^{(2)}}}\sum_{n=1}^{M}\theta_n\delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}_n),\tag{62}$$

where $R^{(2)}$ is the two-dimensional scalar curvature.

The metric of form (59) is still invariant under coordinate transformations in the x,y plane. Using this residual symmetry, we fix the conformal gauge on the plane (this is always possible locally)

$$g_{\alpha\beta}=e^{2\phi}\delta_{\alpha\beta},$$

where $\phi(x,y)$ is some function.

In the conformal gauge, Eqn (61) becomes

$$\partial_\alpha\partial_\beta N=0.$$

For constant boundary conditions for N on the boundary of the x,y plane this equation has the unique solution $N=\mathsf{const}$. Changing the scale of z coordinate, we can set $N=1$ without loss of generality. Then Eqn (62) reduces to the Poisson equation

$$\triangle\phi=-2\pi\sum_n\theta_n\delta^{(2)}(\boldsymbol{r}-\boldsymbol{r}_n),$$

which has the general solution

$$\phi = \sum_n \theta_n \ln |\boldsymbol{r} - \boldsymbol{r}_n| + \frac{1}{2} \ln C, \qquad C = \text{ const} > 0.$$

Thus the metric in the x, y plane is

$$dl^2 = C \prod_n |\boldsymbol{r} - \boldsymbol{r}_n|^{2\theta_n} (dr^2 + r^2 d\varphi^2), \quad 0 \le r < \infty, \;\; 0 \le \varphi < 2\pi, \tag{63}$$

where the polar coordinates r, φ cover the whole plane $\mathbb{R}^2$ and not more (this is important !). Any solution of the Einstein equations is defined up to choosing the coordinate system because the equations are covariant. Using this, we set $C = 1$, which is always possible by choosing the scale of r.

This is indeed the exact solution of the Einstein equations describing arbitrary distribution of parallel wedge dislocations. This statement is made clear from the following consideration.

We note that transition to a continuous distribution of dislocations in the geometric approach is simple. For this, we have to substitute a continuous distribution of sources in the right hand side of the Einstein equations instead of δ-sources.

We consider one wedge dislocation with the source at the origin in more detail to show that metric (63) indeed describes an arbitrary distribution of wedge dislocations

$$T_{zz} = \frac{2\pi}{\sqrt{g^{(2)}}} \theta \delta^{(2)}(x, y). \tag{64}$$

The corresponding metric (63) for $C = 1$ is

$$dl^2 = r^{2\theta} (dr^2 + r^2 d\varphi^2). \tag{65}$$

We pass to a new coordinate system

$$r' = \frac{1}{\alpha} r^\alpha, \qquad \varphi' = \alpha \varphi, \qquad \alpha = 1 + \theta, \tag{66}$$

in which the metric becomes Euclidean

$$dl^2 = dr'^2 + r'^2 d\varphi'^2, \tag{67}$$

but the range of the polar angle differs now from 2π: $0 \le \varphi' < 2\pi\alpha$, and covers the x, y plane with removed or added angle $2\pi\theta$.

Because the metric coincides with the Euclidean one in the primed coordinate system r', φ', we have the Euclidean plane with a removed or added wedge because the angle φ' varies within the interval $(0, 2\pi\alpha)$. The transformation to coordinates r, φ in (66) means the gluing of the edges of the wedge that has appeared, which produces a cone. Therefore, both metrics (65) and (67) describe the same geometric object – conical singularity. The torsion and curvature tensors are obviously zero everywhere except at the origin.

Creation of a conical singularity coincides exactly with creation of the wedge dislocation in the geometric theory of defects. It is not difficult to show that general solution (63)

describes an arbitrary distribution of conical singularities with deficit angles θ_n at points $\boldsymbol{r}_n$. Thus, this solution describes an arbitrary distribution of parallel wedge dislocations.

In the next section, we consider one wedge dislocation in detail. For this, we perform one more coordinate transformation

$$f = \alpha r', \qquad \varphi = \frac{1}{\alpha}\varphi'. \tag{68}$$

Metric (67) then becomes

$$dl^2 = \frac{1}{\alpha^2}df^2 + f^2 d\varphi^2, \qquad \alpha = 1 + \theta. \tag{69}$$

This is one more frequently used form of the metric for a conical singularity.

11 Wedge dislocation in the geometric approach

We now consider a wedge dislocation from the geometric standpoint. From the qualitative standpoint, the creation of a wedge dislocation coincides with the definition of conical singularity. However, there is a quantitative difference because metric (69) depends only on the deficit angle θ and cannot coincide with the induced metric (53) obtained within the elasticity theory. The difference arises because we require the displacement vector in the equilibrium to satisfy equilibrium equations after removing the wedge and gluing its edges (creating conical singularity) in the elasticity theory. At the same time the x, y plane for a conical singularity after the gluing can be deformed in an arbitrary way. Formally, this manifests itself in that metric (53) obtained within the elasticity depends explicitly on the Poisson ratio, which is absent in gravity theory.

We proposed the elastic gauge for solving this problem [23]. We choose elastic gauge (33) as the simplest one for a wedge dislocation. This problem can be solved in two ways. First, the gauge condition can be inserted in the Einstein equations directly. Second, we can find the solution in any suitable coordinate system and then find the coordinate transformation such that the gauge condition is satisfied.

It is easier to follow the second way because the solution for the metric is known, Eqn (69). The vielbein can be associated with metric (69) as

$$e_r{}^{\hat{r}} = \frac{1}{\alpha}, \qquad e_\varphi{}^{\hat{\varphi}} = f.$$

Here, a hat over an index means that it corresponds to the orthonormal coordinate system, and an index without a hat is the coordinate one. Components of this vielbein are the square roots of the corresponding metric components and therefore have the symmetric linear approximation (31). We transform the radial coordinate $f \to f(r)$ because a wedge dislocation is symmetric under rotations in the x, y plane. After that transformation the vielbein components take the form

$$e_r{}^{\hat{r}} = \frac{f'}{\alpha}, \qquad e_\varphi{}^{\hat{\varphi}} = f, \tag{70}$$

where the prime denotes differentiation with respect to r. We choose the vielbein corresponding to the Euclidean metric as

$$\overset{\circ}{e}_r{}^{\hat{r}} = 1, \qquad \overset{\circ}{e}_\varphi{}^{\hat{\varphi}} = r. \tag{71}$$

It defines the Christoffel symbols $\overset{\circ}{\Gamma}_{\mu\nu}{}^{\rho}$ and $\mathbb{SO}(3)$ connection $\overset{\circ}{\omega}_{\mu i}{}^{j}$, which define the covariant derivative. We write only nontrivial components

$$\overset{\circ}{\Gamma}_{r\varphi}{}^{\varphi} = \overset{\circ}{\Gamma}_{\varphi r}{}^{\varphi} = \frac{1}{r}, \qquad \overset{\circ}{\Gamma}_{\varphi\varphi}{}^{r} = -r,$$
$$\overset{\circ}{\omega}_{\varphi\hat{r}}{}^{\hat{\varphi}} = -\overset{\circ}{\omega}_{\varphi\hat{\varphi}}{}^{\hat{r}} = 1.$$

Substitution of the vielbein into gauge condition (33) yields the Euler differential equation for the transition function

$$\frac{f''}{\alpha} + \frac{f'}{\alpha r} - \frac{f}{r^2} + \frac{\sigma}{1-2\sigma}\left(\frac{f''}{\alpha} + \frac{f'}{r} - \frac{f}{r^2}\right) = 0.$$

Its general solution depends on two arbitrary constants $C_{1,2}$,

$$f = C_1 r^{\gamma_1} + C_2 r^{\gamma_2}, \tag{72}$$

where the exponents $\gamma_{1,2}$ are defined by the quadratic equation

$$\gamma^2 + 2\gamma\theta b - \alpha = 0, \qquad b = \frac{\sigma}{2(1-\sigma)},$$

which has real roots for $\theta > -1$ with different signs: positive root γ_1 and negative γ_2. We recall that there are thermodynamical constraints $-1 \le \sigma \le 1/2$ on the Poisson ratio [61].

To fix the constants, we impose boundary conditions on the vielbein:

$$e_r{}^{\hat{r}}\big|_{r=R} = 1, \qquad e_\varphi{}^{\hat{\varphi}}\big|_{r=0} = 0. \tag{73}$$

The first boundary condition corresponds to the last boundary condition for displacement vector (50) (the absence of external forces on the surface of the cylinder), and the second one corresponds to the absence of the angular component of the deformation tensor at the core of dislocation. Equations (73) define the values of integration constants

$$C_1 = \frac{\alpha}{\gamma_1 R^{\gamma_1 - 1}}, \qquad C_2 = 0. \tag{74}$$

The obtained vielbein defines the metric

$$dl^2 = \left(\frac{r}{R}\right)^{2\gamma_1 - 2}\left(dr^2 + \frac{\alpha^2 r^2}{\gamma_1^2} d\varphi^2\right), \tag{75}$$

where

$$\gamma_1 = -\theta b + \sqrt{\theta^2 b^2 + 1 + \theta}.$$

This is the solution of the posed problem. The derived solution is valid for all deficit angles θ and for all $0 < r < R$. The obtained metric depends on three constants: θ, σ, and R. The dependence on the deficit angle θ is due to its occurrence on the right hand side of Einstein equations (58). The dependence on the Poisson ratio comes from elastic gauge (33), and, finally, the dependence on the cylinder radius comes from boundary condition (73).

If a wedge dislocation is absent, then $\theta = 0$, $\alpha = 1$, $\gamma_1 = 1$, and metric (75) goes to the Euclidean one $dl^2 = dr^2 + r^2 d\varphi^2$, as expected.

We compare metric (75) obtained within the geometric approach with the induced metric from elasticity theory in Eqn (53). First, it has a simpler form. Second, in the linear approximation in θ

$$\gamma_1 \approx 1 + \theta \frac{1 - 2\sigma}{2(1 - \sigma)},$$

and metric (75) can be easily shown to coincide precisely with metric (53) obtained within the elasticity theory. We see that induced metric (53) provides only the linear approximation for the metric obtained within the geometric theory of defects, which, in addition, has a simpler form. Beyond the perturbation theory, we see essential differences. In particular, metric (53) is singular at the origin, whereas metric (75) obtained beyound the perturbation theory is regular.

The stress and deformation tensors are related by Hook's law (3). There is an experimental possibility to check formulas (75) because the deformation tensor is the linear approximation for the induced metric. For this one has to measure the stress field for a single wedge dislocation. In this way, the geometric theory of defects can be experimentally confirmed or discarded.

The problem of reconstruction of the displacement field for a given metric reduces to solution of differential equations (5) with metric (75) on the right hand side and boundary conditions (50). We do not discuss this problem here. We note that in the geometric theory of defects, a complicated stage of finding the displacement vector where it exists is simply absent and is not necessary.

Two-dimensional metric (75) describes the conical singularity in the elastic gauge. The relation between conical singularities and wedge dislocations was established in [22,78–80]. In these papers, the metric was used in the other gauges (coordinate systems).

12 Elastic oscillations in media with dislocations

Elastic oscillations in elastic media without defects are described by a time-dependent vector field $u^i(t, x)$ that satisfies the wave equation (see, e.g., [61])

$$\rho_0 \ddot{u}^i - \mu \triangle u^i - (\lambda + \mu) \partial^i \partial_j u^j = 0, \tag{76}$$

where dots denote differentiation with respect to time and ρ_0 is the mass density of medium, which is assumed to be constant. If the medium contains defects, then the metric of the space becomes nontrivial, $\delta_{ij} \to g_{\mu\nu} = e_\mu{}^i e_\nu{}^j \delta_{ij}$. We assume that relative displacements for elastic oscillations are much smaller than stresses induced by defects:

$$\partial_\mu u^i \ll e_\mu{}^i. \tag{77}$$

Then, in the first approximation, we assume that elastic oscillations propagate in a Riemannian space with a nontrivial metric induced by dislocations. Here, we discard changes in the metric due to elastic oscillation themselves. Therefore, for elastic oscillations we postulate the following equation, which is a covariant generalization of (76) for spatial variables:

$$\rho_0 \ddot{u}^i - \mu \tilde{\triangle} u^i - (\lambda + \mu) \widetilde{\nabla}^i \widetilde{\nabla}_j u^j = 0, \tag{78}$$

where u^i denote components of displacement vector field with respect to an orthonormal space basis e_i (see the Appendix), $\tilde{\triangle} = \widetilde{\nabla}^i\widetilde{\nabla}_i$ is the covariant Laplace–Beltrami operator built for the vielbein ${e_\mu}^i$, and $\widetilde{\nabla}_i$ is the covariant derivative. The explicit form of the covariant derivative of the displacement field is

$$\widetilde{\nabla}_i u^j = {e^\mu}_i \widetilde{\nabla}_\mu u^j = {e^\mu}_i(\partial_\mu u^j + u^k {\widetilde{\omega}_{\mu k}}{}^j),$$

where ${\widetilde{\omega}_{\mu k}}{}^j$ is the $\mathbb{SO}(3)$ connection built for zero torsion.

We note that the displacement vector field describing elastic oscillations is not the total displacement vector field of points of a medium with dislocations. It was already said in section 3 that the displacement field for dislocations can be introduced only in those regions of media where defects are absent. If we denote it by u^i_{D}, then the total displacement field in these regions is defined by the sum $u^i_{\mathrm{D}} + u^i$. There, the vielbein is defined only by the displacement field for dislocations ${e_\mu}^i = \partial_\mu u^i_{\mathrm{D}}$. We note that the smallness of relative deformations (77) is also meaningful in those regions of space where displacements u^i_{D} are not defined.

We now decompose the displacement field covariantly into transversal $u^{\mathrm{T}i}$ and longitudinal parts $u^{\mathrm{L}i}$,

$$u^i = u^{\mathrm{T}i} + u^{\mathrm{L}i},$$

which are defined by the relations

$$\widetilde{\nabla}_i u^{\mathrm{T}i} = 0, \tag{79}$$

$$\widetilde{\nabla}_i u^{\mathrm{L}}_j - \widetilde{\nabla}_j u^{\mathrm{L}}_i = 0. \tag{80}$$

The decomposition of a vector field into longitudinal and transversal parts in three-dimensional space with an accuracy up a constant is unique. We recall that Latin indices are lowered with the help of Kronecker symbols, $u^i = u_i$, and this operation commutes with covariant differentiation. Equation (80) can be rewritten as

$$\widetilde{\nabla}_i u^{\mathrm{L}}_j - \widetilde{\nabla}_j u^{\mathrm{L}}_i = {e^\mu}_i {e^\nu}_j(\widetilde{\nabla}_\mu u^{\mathrm{L}}_\nu - \widetilde{\nabla}_\nu u^{\mathrm{L}}_\mu) = {e^\mu}_i {e^\nu}_j(\partial_\mu u^{\mathrm{L}}_\nu - \partial_\nu u^{\mathrm{L}}_\mu) = 0,$$

because transformation of Latin indices into Greek ones commutes with the covariant differentiation, and the Christoffel symbols are symmetrical in the first two indices. The last equality means that the 1-form $dx^\mu u^{\mathrm{L}}_\mu$ is closed. It is easily confirmed that Eqn (78) for elastic oscillations is equivalent to two independent equations for transverse and longitudinal oscillations,

$$\frac{1}{c^2_{\mathrm{T}}}\ddot{u}^{\mathrm{T}i} - \tilde{\triangle} u^{\mathrm{T}i} = 0, \qquad \frac{1}{c^2_{\mathrm{L}}}\ddot{u}^{\mathrm{L}i} - \tilde{\triangle} u^{\mathrm{T}i} = 0, \tag{81}$$

where

$$c^2_{\mathrm{T}} = \frac{\mu}{\rho_0}, \qquad c^2_{\mathrm{L}} = \frac{\lambda + 2\mu}{\rho_0}$$

are the squares of sound velocities for transverse and longitudinal oscillations.

Particles arising after the secondary quantization of Eqns (81) are called phonons in solids. Therefore, strictly speaking, the problem of scattering of phonons on dislocations is a quantum mechanical one. In the present review, we consider only classical aspects of this problem.

Wave equations (81) contain second and first derivatives of the displacement field. The latter are contained in the covariant Laplace–Beltrami operator $\tilde{\triangle}$. The terms with second derivatives can be written in the four-dimensional form $g^{\alpha\beta}\partial_\alpha\partial_\beta$ where $g^{\alpha\beta}$ is the inverse metric to

$$g_{\alpha\beta} = \begin{pmatrix} c^2 & 0 \\ 0 & -g_{\mu\nu} \end{pmatrix}, \tag{82}$$

where c is either the transverse or longitudinal sound velocity. Above, we used the following notations. Four-dimensional coordinates are denoted by Greek letters from the beginning of the alphabet $\{x^\alpha\} = \{x^0 = t, x^1, x^2, x^3\}$, and letters from the middle of Greek alphabet denote only space coordinates $\{x^\mu\} = \{x^1, x^2, x^3\}$. This rule can be easily remembered by the following inclusions $\{1, 2, 3\} \subset \{0, 1, 2, 3\}$ and $\{\mu, \nu, \dots\} \subset \{\alpha, \beta, \dots\}$. Christoffel symbols (115) can be computed for four-dimensional metric (82), which defines a system of ordinary nonlinear equations for extremals $x^\alpha(\tau)$ (lines of extremal lengths that coincide with geodesics in Riemannian geometry), where dots denote differentiation with respect to canonical parameter τ. For the block-diagonal metric in (82), these equations decompose:

$$\ddot{x}^0 = 0, \tag{83}$$

$$\ddot{x}^\mu = -\widetilde{\Gamma}_{\nu\rho}{}^\mu \dot{x}^\nu \dot{x}^\rho, \tag{84}$$

where $\widetilde{\Gamma}_{\nu\rho}{}^\mu$ are the three-dimensional Christoffel symbols constructed for the three-dimensional metric $g_{\mu\nu}$ which, as we recall, depends only on spatial coordinates for a static distribution of defects. Let $\{x^\alpha(\tau)\}$ be an arbitrary extremal in four-dimensional space-time. For metric (82), its natural projection on space $\{x^\alpha(\tau)\} \to \{0, x^\mu(\tau)\}$ is also an extremal but now for the spatial part of the metric $g_{\mu\nu}$.

Equations for extremals (83), (84) are invariant under linear transformations of the canonical parameter τ. Therefore, the canonical parameter can be identified with time, $\tau = t = x^0$, without loss of generality as a consequence of Eqn (83).

We assume that a particle moves in space along an extremal $x^\mu(t)$ with velocity v. This means that

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = v^2.$$

The length of the tangent vector to the corresponding extremal $\{t, x^\mu(t)\}$ in four-dimensional space-time is then equal to

$$g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = c^2 - v^2.$$

Hence, if the particle moves in space along extremal with velocity less than, equal to, or greater than the speed of sound, then its world line in space-time is timelike, null, or spacelike, respectively.

We return now to propagation of phonons in media with defects. As in geometric optics [81], there are useful notions of wave fronts and rays in the analysis of the asymptotic form of solutions for wave equations (81). We do not consider mathematical aspects of this approach, which is nontrivial and complicated [82], and instead give only a physical description. In the eikonal (high frequency) approximation, phonons propagate along rays coinciding with null extremals for the four-dimensional metric $g_{\alpha\beta}$. Forms of rays, which are identified with trajectories of phonons, are defined by the three-dimensional metric $g_{\mu\nu}$. This means that in the eikonal approximation, trajectories of transverse and

longitudinal phonons in a medium with defects are the same and are defined by Eqn (84). The difference reduces to the velocities of propagation for transverse and longitudinal phonons being different and equal to c_{T} and c_{L}, respectively.

13 Scattering of phonons on a wedge dislocation

Calculations in the present section coincide, in fact, with the analysis performed in Section 3 of [59]. The difference is that in what follows, we use the metric written in the elastic gauge. This is important because we assume that trajectories of phonons seen in an experiment coincide with extremals for the metric precisely in this gauge.

In the presence of one wedge dislocation, the space metric in the cylindrical coordinates r, φ, z is

$$g_{\mu\nu} = \begin{pmatrix} \dfrac{r^{2\gamma-2}}{R^{2\gamma-2}} & 0 & 0 \\ 0 & \dfrac{\alpha^2}{\gamma^2}\dfrac{r^{2\gamma}}{R^{2\gamma-2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{85}$$

where the nontrivial part of the metric in the r, φ plane was obtain earlier [see Eqn (75)]. Here, we change γ_1 to γ for simplicity of notation. The inverse metric is also diagonal,

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{R^{2\gamma-2}}{r^{2\gamma-2}} & 0 & 0 \\ 0 & \dfrac{\gamma^2}{\alpha^2}\dfrac{R^{2\gamma-2}}{r^{2\gamma}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Christoffel symbols for metric (85) are calculated according to formulas (115). As a result, only four Christoffel symbols differ from zero:

$$\begin{aligned} \widetilde{\Gamma}_{rr}{}^{r} &= \frac{\gamma-1}{r}, \\ \widetilde{\Gamma}_{\varphi\varphi}{}^{r} &= -\frac{\alpha^2 r}{\gamma}, \\ \widetilde{\Gamma}_{r\varphi}{}^{\varphi} &= \widetilde{\Gamma}_{\varphi r}{}^{\varphi} = \frac{\gamma}{r}. \end{aligned}$$

In the preceeding section, we showed that in the eikonal approximation, phonons propagate along extremals $x^\mu(t)$ defined by Eqns (84). In the case considered here, these equations are

$$\ddot{r} = -\frac{\gamma-1}{r}\dot{r}^2 + \frac{\alpha^2}{\gamma}r\dot{\varphi}^2, \tag{86}$$

$$\ddot{\varphi} = -\frac{2\gamma}{r}\dot{r}\dot{\varphi}, \tag{87}$$

$$\ddot{z} = 0, \tag{88}$$

where the dot denotes differentiation with respect to time t. It follows from the last equation that phonons move along the z axis with constant velocity, which corresponds

to translational invariance along z. This means that scattering on a wedge dislocation is reduced to a two-dimensional problem in the r, φ plane, as could be expected.

The system of equations for $r(t)$ and $\varphi(t)$ in (86) and (87) can be explicitly integrated. For this, we find two first integrals. First, for any metric, there is the integral for equations for extremals

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = \mathsf{const}.$$

We then have the equality

$$r^{2\gamma-2}\dot{r}^2 + \frac{\alpha^2}{\gamma^2}r^{2\gamma}\dot{\varphi}^2 = C_0 = \mathsf{const} > 0. \tag{89}$$

Second, the invariance of the metric under rotations about the z axis results in the existence of an additional integral. It is constructed as follows. There is a Killing vector corresponding to the invariance of the metric, which in cylindrical coordinates has the simple form $k = \partial_{\varphi}$. Straightforward verification proves that

$$g_{\mu\nu}k^{\mu}\dot{x}^{\nu} = \mathsf{const}.$$

In the considered case, this results in the identity

$$r^{2\gamma}\dot{\varphi} = C_1 = \mathsf{const}. \tag{90}$$

We analyze the form of an extremal $r = r(\varphi)$. First derivatives can be found from Eqns (89) and (90)

$$\dot{r} = \pm r^{-2\gamma+1}\sqrt{C_0 r^{2\gamma} - \frac{\alpha^2}{\gamma^2}C_1^2}, \tag{91}$$

$$\dot{\varphi} = C_1 r^{-2\gamma}. \tag{92}$$

Admissible values of the radial coordinate r for which the expression under the square root is nonnegative are to be found later. From the above equations, we obtain the equation defining the form of nonradial ($C_1 \neq 0$) extremals

$$\frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}} = \pm r\sqrt{\frac{C_0}{C_1^2}r^{2\gamma} - \frac{\alpha^2}{\gamma^2}}. \tag{93}$$

This equation can be easily integrated, and we finally obtain explicit formulas defining the form of an extremal:

$$\left(\frac{r}{r_m}\right)^{2\gamma} \sin^2[\alpha(\varphi + \varphi_0)] = 1, \tag{94}$$

where

$$r_m = \left(\frac{C_1\alpha}{\sqrt{C_0}\gamma}\right)^{1/\gamma} = \mathsf{const} > 0, \qquad \varphi_0 = \mathsf{const}.$$

The constant r_m is positive and defines the minimal distance at which an extremal approaches the core of the dislocation, i.e., $r \geq r_m$. Only for these values of r is the

expression under the root in Eqn (93) nonnegative. The integration constant φ_0 is arbitrary and corresponds to the invariance of the problem under rotations around the core of dislocation.

Equations for extremals (86) and (87) also have degenerate solutions:

$$\frac{1}{\gamma}r^{\gamma} = \pm\sqrt{C_0}(t+t_0), \qquad \varphi = \mathsf{const}, \qquad t_0 = \mathsf{const}. \tag{95}$$

These extremals correspond to the radial motion of phonons. Such trajectories are unstable in the sense that there are no nonradial extremals near them.

We note that circular extremals $r = \mathsf{const}$ are absent as a consequence of Eqn (86), although integrals of motion (89) and (90) admit such a solution. This is because Eqn (86) was multiplied by $\dot{r}$ in deriving first integral (89).

We now analyze the form of nonradial extremals (94). For any extremal, the radius r decreases first from infinity to the minimal value r_m and then increases from r_m to infinity. We can assume here without loss of generality that the argument of sine in (94) varies from 0 to π. Thus, we obtain the range of changes of the polar angle

$$0 \le \varphi + \varphi_0 \le \frac{\pi}{\alpha}.$$

This means that the extremal comes from infinity at the angle $-\varphi_0$ and goes to infinity at the angle $\pi/\alpha - \varphi_0$. This corresponds to the scattering angle

$$\chi = \pi - \frac{\pi}{\alpha} = \frac{\pi\theta}{1+\theta}. \tag{96}$$

We note that the scattering angle depends only on the deficit angle θ and does not depend on the elastic properties of the medium. The scattering angle has a simple physical interpretation. For positive θ, the medium is cut and moved apart. A wedge of the same medium without elastic stresses is inserted in the cavity that has appeared. Afterwards, gluing is performed, and the wedge is compressed. The compression coefficient for all circles centered at the origin is equal to $1/(1+\theta)$ due to symmetry considerations. Therefore, the scattering angle equals half the deficit angle times the compression coefficient,

$$\chi = \frac{2\pi\theta}{2} \times \frac{1}{1+\theta}.$$

For $\theta = 0$, the dislocation is absent, and the scattering angle is equal to zero.

For positive deficit angles, the scattering angle is positive, which corresponds to repulsion of phonons from the wedge dislocation. Corresponding extremals are shown in Fig.11a, and they have asymptotes as $r \to \infty$. We note that for positive deficit angles, no two points on the r, φ plane can be connected by an extremal, i.e., there is a domain to the right of the wedge dislocation that cannot be reached at all by a phonon falling from the left.

All extremals shown in the figures in the present section are calculated numerically. The values of the deficit angle θ, the scattering angle χ, and the minimal distance r_m to the dislocation axis are shown in the figures. For definiteness, we choose $\varphi_0 = \pi$ corresponding to the fall of phonons from the left.

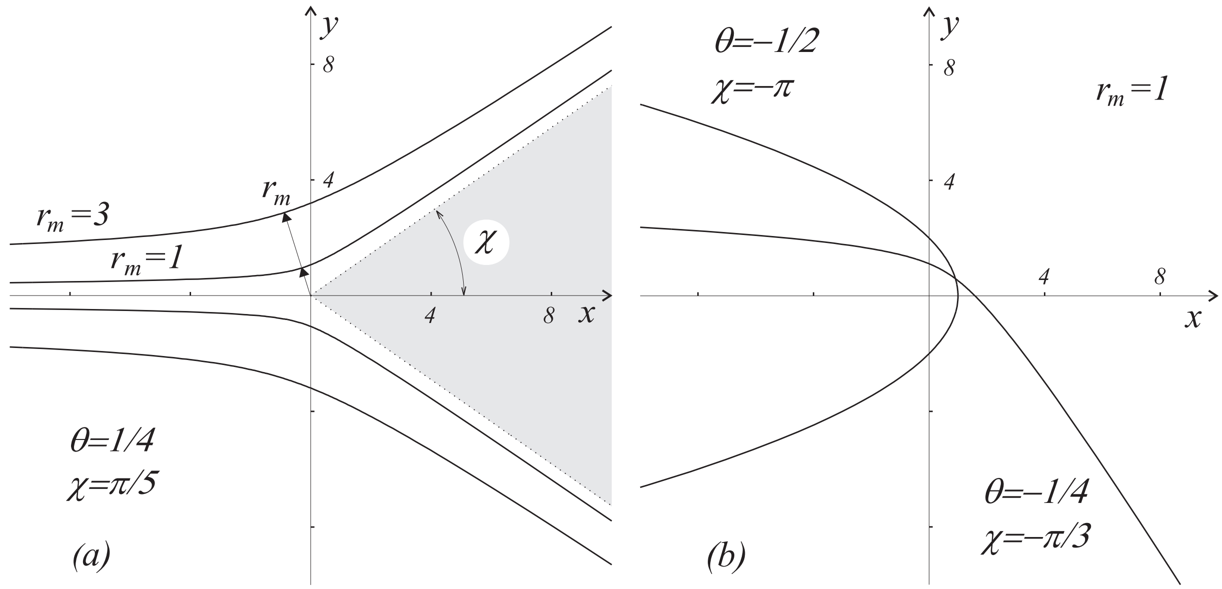

Figure 11: *(a)* Extremals for the wedge dislocation with a positive ($\theta > 0$) deficit angle. Two extremals and their reflections with respect to the x axis are shown for the same $\theta > 0$ but different r_m. *(b)* Extremals for the wedge dislocation with a negative ($-1/2 \le \theta < 0$) deficit angle. Two extremals are shown for the same r_m but different deficit angles θ.

For negative deficit angles the scattering angle is negative and is defined by the same formulas (96). This corresponds to the attraction of phonons to the dislocation axis. In Fig.11b, we show two extremals with the same parameter r_m but for two dislocations with different deficit angles. For $-1/2 < \theta < 0$, the scattering angle varies from 0 to 2π (Fig.11b). For $\theta = -1/2$, the scattering angle is equal to 2π. We note that for negative deficit angles, phonons have no asymptotes as $r \to \infty$, i.e., for $\theta = -1/2$, phonons fall from infinity ($x \to -\infty, y \to +\infty$) and return to the infinity ($x \to -\infty, y \to -\infty$).

When the deficit angle is sufficiently small ($-1 < \theta < -1/2$), a phonon makes one or several rotations around the dislocation and then goes to infinity. Examples of such trajectories are shown in Figs. 12–14.

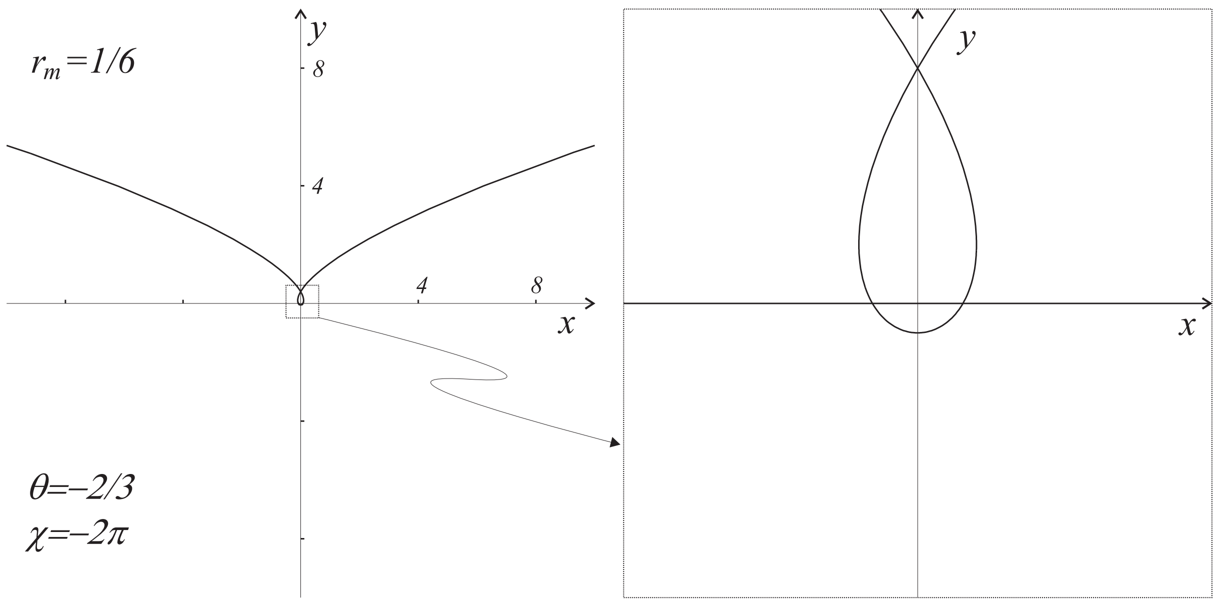

Figure 12: For $\theta = -2/3$, an extremal makes one rotation around the dislocation and then goes forward in the original direction. The blown-up part of the trajectory is shown in the square to the right.

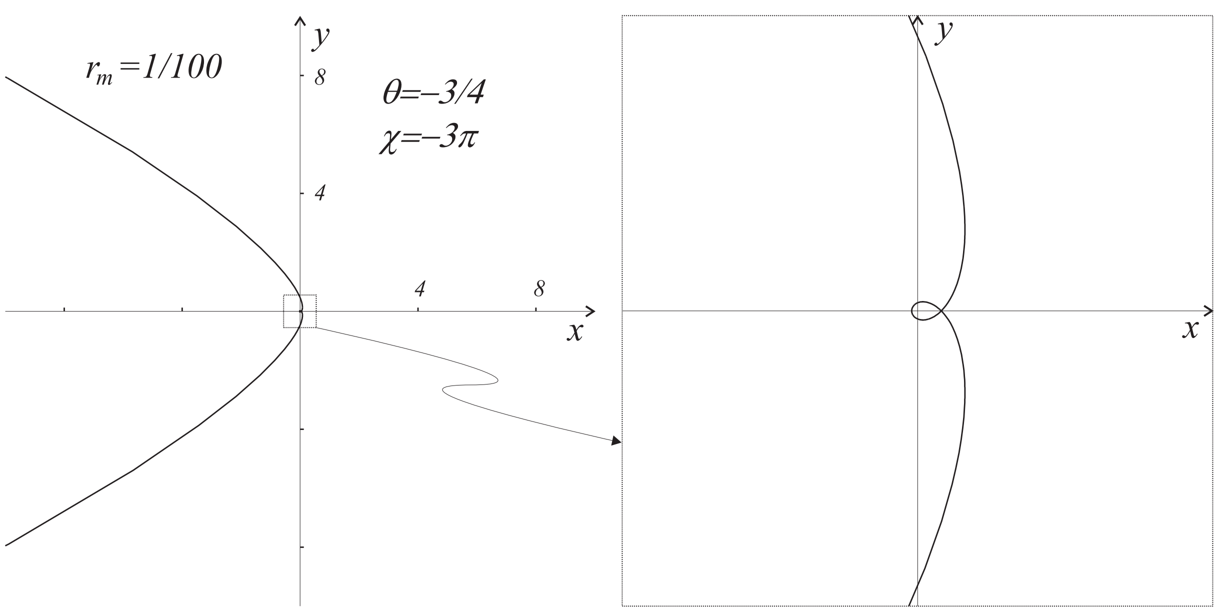

Figure 13: For $\theta = -3/4$, an extremal makes two turns around the dislocation and then goes back. The blown-up part of the trajectory is shown in the square to the right.

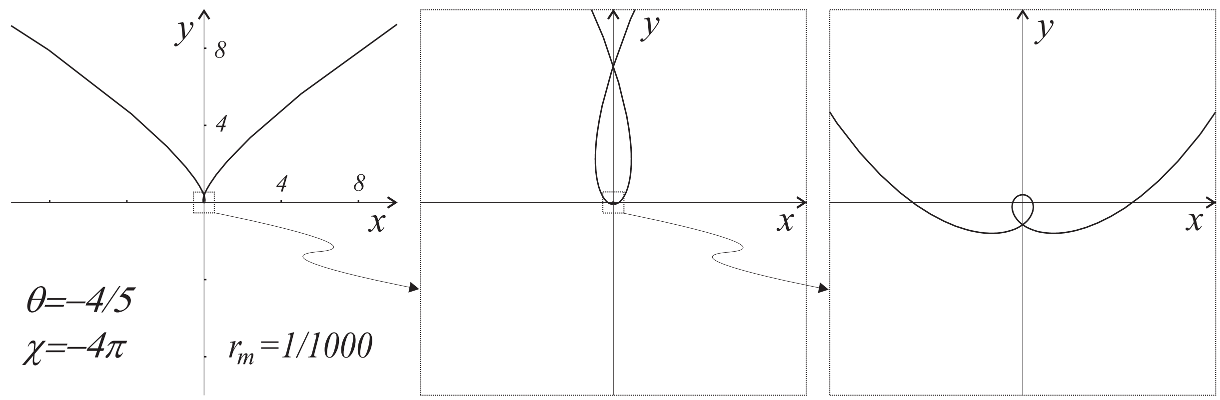

Figure 14: For $\theta = -4/5$, an extremal makes two and a half rotations around the dislocation and then goes forward in the original direction. Subsequent blown-up parts of the trajectory are shown in the two squares to the right.

We consider the asymptotic behavior of nonradial extremals as $r \to \infty$. As a consequence of Eqn (91), far from the core of dislocation, we obtain

$$\dot{r} \approx \pm\sqrt{C_0} r^{-\gamma+1}.$$

It follows from this equation that the dependence of the radius on time is the same as for radial extremals (95). Because $\gamma > 0$, an infinite value of r corresponds to an infinite value of time t. This means that the r, φ plane with the given metric is complete at $r \to \infty$. The origin (the core of dislocation) is a singular point. Only radial extremals fall into it at a finite moment of time.

Integrals of motion (89) and (90) have simple physical meaning. Equations for extremals (84) follow from the variational principle for the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \tag{97}$$

describing motion of a free massless point particle in a nontrivial metric $g_{\mu\nu}(x)$. Here, the metric is considered as a given external field and is not varied.

The energy corresponding to this integral is equal to

$$E = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \frac{r^{2\gamma-2}}{R^{2\gamma-2}} \dot{r}^2 + \frac{1}{2} \frac{\alpha^2}{\gamma^2} \frac{r^{2\gamma}}{R^{2\gamma-2}} \dot{\varphi}^2 + \frac{1}{2} \dot{z}^2.$$

If the metric, as in our case, does not depend on time explicitly, then the energy is conserved ($E = \mathsf{const}$) and its numerical value for the motion in the r, φ plane is proportional to the integral of motion C_0.

For a wedge dislocation, the metric is independent on the polar angle φ, and the Lagrangian is invariant under rotations: $\varphi \to \varphi + \mathsf{const}$. By the Noether theorem, the angular momentum conservation law corresponds to this invariance,

$$J = -\frac{\alpha^2}{\gamma^2} \frac{r^{2\gamma}}{R^{2\gamma-2}} \dot{\varphi} = \mathsf{const}.$$

As a consequence, the constant of integration C_1 is proportional to the angular momentum.

We note that the behavior of extremals differs qualitatively from trajectories of point particles moving in flat space with Euclidean metric $\delta_{\mu\nu}$ in an external potential field $U(x)$. It can be easily shown that trajectories of point particles described by the Lagrangian

$$L = \frac{1}{2}\delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu - U$$

cannot coincide with extremals (84) for any function $U(x)$.

In the present section, we have showed how the problem of scattering of phonons on the simplest wedge dislocation is solved in the geometric approach. The results of calculations can be checked experimentally. The more general problem of scattering of phonons on an arbitrary distribution of wedge dislocations, including the edge dislocation and continuous distribution of dislocations, is solved in [59]. Solution of this problem in the geometric approach is simpler than in the framework of classical elasticity theory, where we have to solve partial differential equations with complicated boundary conditions. In [59], calculations were performed in the conformal gauge for the metric. For comparison with experiment, the results must be recalculated in the elastic gauge, which has physical meaning in the geometric approach.

14 An impurity in the field of a wedge dislocation

We consider elastic media with one wedge dislocation containing one atom of impurity or vacancy. If we consider the influence of impurity on the distribution of elastic stresses small compared with the elastic stresses induced by the dislocation itself, then the motion of the impurity can be considered as taking place in three-dimensional space with the nontrivial metric (75). In the geometric approach, we assume that the potential energy of the interaction equals zero, and all interactions are due to changes in the kinetic energy, which depends explicitly on the nontrivial metric.

We solve the corresponding quantum mechanical problem. We consider bounded states of impurity moving inside a cylinder of radius R in the presence of a wedge dislocation. We assume that the cylinder axis coincides with the core of dislocation. The stationary Schrödinger equation is

$$-\frac{\hbar^2}{2M}\tilde{\triangle}\Psi = E\Psi, \tag{98}$$

where $\hbar$ is the Plank constant and M, Ψ, and E are mass, wave function, and energy of the impurity. The nontriviality of the interaction with dislocation reduces to the nontrivial Laplace–Beltrami operator

$$\tilde{\triangle}\Psi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\Psi),$$

where the metric was found earlier [see Eqn (75)], and $g = \mathsf{det}\, g_{\mu\nu}$.

Taking the symmetry of the problem into account, we solve the Schrödinger equation (98) in cylindrical coordinates by the separation of variables. Let

$$\Psi(r,\varphi,z) = Z(z)\sum_{m=-\infty}^{\infty}\psi_m(r)\,\mathsf{e}^{im\varphi},$$

where we have the two following possibilities for the normalization function $Z(z)$. If the impurity moves freely along the z axis with momentum $\hbar k$, then

$$Z(z) = \frac{1}{\sqrt{2\pi}} \mathsf{e}^{ikz}.$$

If its motion is restricted by the planes $z = 0$ and $z = z_0$, then

$$Z(z) = \sqrt{\frac{2}{z_0}} \sin(k_l z), \qquad k_l = \frac{\pi l}{z_0}.$$

In what follows, we drop the integer-valued subscript l indicating the restricted motion, having both possibilities in mind.

The condition for the constant m (the eigenvalue of the projection of the momentum on the z axis) to be an integer appears due to the periodicity condition

$$\Psi(r, \varphi, z) = \Psi(r, \varphi + 2\pi, z).$$

Then for the radial wave function, $\psi_m(r)$ we then obtain the equation

$$\frac{R^{2\gamma-2}}{r^{2\gamma-1}} \partial_r (r \partial_r \psi_m) + \left(\frac{2ME}{\hbar^2} - \frac{\gamma^2}{\alpha^2} \frac{R^{2\gamma-2}}{r^{2\gamma}} m^2 - k^2 \right) \psi_m = 0. \tag{99}$$

We introduce the new radial coordinate

$$\rho = \frac{r^\gamma}{\gamma R^{\gamma-1}}.$$

This is the transformation in (72) and (74) up to a constant. The radial equation is then

$$\frac{1}{\rho} \partial_\rho (\rho \partial_\rho \psi_m) + \left(\beta^2 - \frac{\nu^2}{\rho^2} \right) \psi_m = 0, \tag{100}$$

where

$$\beta^2 = \frac{2ME}{\hbar^2} - k^2, \qquad \nu = \frac{|m|}{\alpha} > 0.$$

This is the Bessel equation. We solve it with the boundary condition

$$\psi_m|_{\rho = R/\gamma} = 0, \tag{101}$$

which corresponds to the motion of an impurity inside the cylinder with an impenetrable boundary. The general solution of the Bessel equation (100) is

$$\psi_m = c_m J_\nu(\beta\rho) + d_m N_\nu(\beta\rho), \qquad c_m, d_m = \text{const},$$

where J_ν and N_ν are Bessel and Neumann functions of order ν [83]. The boundedness of the wave function on the axis of the cylinder requires $d_m = 0$. The constants of integration c_m are found from the normalization condition

$$\int_0^R dr\, r |\psi_m|^2 = 1.$$

Boundary condition (101) yields the equation for β

$$J_\nu(\beta R/\gamma) = 0, \tag{102}$$

which defines the energy levels of bounded states. It is well known that for real $\nu > -1$ and R/γ, this equation has only real roots. Positive roots form an infinite countable set and all of them are simple [83]. This provides the inequality

$$\beta^2 = \frac{2ME}{\hbar^2} - k^2 \geq 0.$$

We label the positive roots of Eqn (102) by the index $n = 1, 2, \ldots$ (principle quantum number): $\beta \to \beta_n(m, \alpha, \gamma, R)$. Then the spectrum of bounded states is

$$E_n = \frac{\hbar^2}{2M}(k^2 + \beta_n^2). \tag{103}$$

For large radii $\beta\rho \gg 1$ and $\beta\rho \gg \nu$ we have the asymptotic form:

$$J_\nu(\beta\rho) \approx \sqrt{\frac{2}{\pi\beta\rho}} \cos\left(\beta\rho - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

As a result, we obtain explicit expression for the spectrum

$$\beta_n = \frac{\gamma\pi}{R}\left(n + \frac{|m|}{2\alpha} - \frac{1}{4}\right). \tag{104}$$

In the absence of defect, $\alpha = 1$, $\gamma = 1$, $\rho = r$, and the radial functions ψ_m are expressed through the Bessel functions of integer order $\nu = |m|$. In this case, the spectrum of bounded states depends only on the sizes of the cylinder. In the presence of the wedge dislocation, Bessel functions have a noninteger order in general. In this case, the spectrum of energy levels acquires dependence on the deficit angle θ and the Poisson ratio σ characterizing elastic properties of the medium.

If the mass of impurity and vacancy is defined by integral (10), then $M > 0$ for impurity and the energy eigenvalues are positive. For vacancy, $M < 0$ and the energy eigenvalues are negative. In this case, the energy spectrum is not bounded from below, which causes serious problems for physical interpretation. It seems that one has to insert in the Schrödinger equation not the bare mass (10) but the effective mass, with the contribution of elastic stresses arising around a vacancy taken into account. This question presently remains unanswered.

The presentation in the present section is close to that in [28]. In contrast to that paper, we use the elastic gauge for the metric. Therefore, our results depend not only on the deficit angle of a wedge dislocation but also on the elastic properties of the medium.

Calculations of the energy levels of an impurity in the field of a wedge dislocation are actually equivalent to the calculations of bounded-state energies in the Aharonov–Bohm effect [84] (see reviews [85, 86]). The difference reduces only to changing the order of the Bessel functions,

$$\nu = \left| m - \frac{\Phi}{\Phi_0} \right|,$$

where $\Phi_0 = 2\pi\hbar c/e$ is the magnetic flux quantum and e is the electron charge.

The considered example shows how the influence of the presence of defects is taken into account in the geometric approach in the first approximation. If calculations in some problem were performed in elastic media without defects, then to take the influence of defects into account we have to replace the flat Euclidean metric with a nontrivial metric describing the given distribution of defects. This problem may appear complicated mathematically because we have to solve the three-dimensional Einstein equations to find the metric. However, there are no principal difficulties: the effect of dislocations reduces to a change in the metric.

15 Conclusion

The geometric theory of defects describes defects in elastic media (dislocations) and defects in the spin structure (disclinations) from a single point of view. This model can be used for describing single defects as well as their continuous distribution. The geometric theory of defects is based on the Riemann-Cartan geometry. By definition, torsion and curvature tensors are equal to surface densities of Burgers and Frank vectors, respectively.

Equations defining the static distribution of defects are covariant and have the same form as equations of gravity models with dynamical torsion. To choose a solution uniquely, one must fix the coordinate system. For this, the elastic gauge for the vielbein and the Lorentz gauge for the $\mathbb{SO}(3)$ connection are proposed. In the defect-free case the displacement vector field and the field of the spin structure can be introduced. Equations of equilibrium are then identically satisfied, and the gauge conditions reduce to the equations of elasticity theory and the principal chiral $\mathbb{SO}(3)$-field. In this way, the geometric theory of defects incorporates elasticity theory and the model of principal chiral field.

In a certain sense, the elastic gauge represents the equations of the nonlinear elasticity theory. Nonlinearity is introduced in elasticity theory in two ways. First, the deformation tensor is defined through the induced metric

$$\epsilon_{ij} = \frac{1}{2}(\delta_{ij} - g_{ij})$$

instead of being defined by linear relation (4). Then the stress tensor is given by an infinite series in the displacement vector. Second, Hook's law can be modified assuming nonlinear dependence of the stress tensor on the deformation tensor. Hence, the elastic gauge condition is the equations of the nonlinear elasticity theory where the deformation tensor is assumed to be defined through the induced metric and Hook's law is kept linear. A generalization to the nonlinear dependence of the deformation tensor on the stress tensor is obvious.

As an example, we considered the wedge dislocation from the standpoint of the elasticity theory and the geometric theory of defects. We showed that the elasticity theory reproduces only the linear approximation of the geometric approach. In contrast to the induced metric obtained within the exact solution of the linear elasticity theory, the expression for the metric obtained as the exact solution of the Einstein equations in the elastic gauge is simpler and is defined on the whole space and for all deficit angles. The obtained expression for the metric can be checked experimentally.

Two problems are considered as applications of the geometric theory of defects. The first is the scattering of phonons on a wedge dislocation. In the eikonal approximation, the problem is reduced to the analysis of extremals for the metric describing a given dislocation. Equations for extremals are integrated explicitly, and the scattering angle is found. The second of the considered problems is the construction of the wave functions and energy spectrum of impurity in the presence of a wedge dislocation. This requires solving the Schrödinger equation. This problem is mathematically equivalent to solving the Schrödinger equation for bound states in the Aharonov–Bohm effect [86]. The explicit dependence of the spectrum on the deficit angle and elastic properties of the medium is found in the presence of a wedge dislocation.

The geometric theory of static distribution of defects can also be constructed for membranes, i.e., on the plane $\mathbb{R}^2$. For this, one has to consider the Euclidean version [87] of two-dimensional gravity with torsion [88–90]. This model is favored by its integrability [91–94].

The developed geometric construction in the theory of defects can be inverted, and we can consider the gravitational interaction of masses in the universe as the interaction of defects in elastic ether. Point masses and cosmic strings [95, 96] then correspond to point defects (vacancies and impurities) and wedge dislocations. In this interpretation of gravity, we have a question about the elastic gauge, which has direct physical meaning in the geometric theory of defects. If we take the standpoint of the theory of defects, then the elastic properties of ether correspond to some value of the Poisson ratio, which can be measured experimentally.

It seems interesting and important for applications to include time in the considered static approach for describing motion of defects in the medium. Such a model is lacking at present. From the geometric standpoint this generalization can be easily performed, at least in principle. It is sufficient to change the Euclidean space $\mathbb{R}^3$ to the Minkowski space $\mathbb{R}^{1,3}$ and to write a suitable Lagrangian quadratic in curvature and torsion, which corresponds to the true gravity model with torsion. One of the arising problems is the physical interpretation of the additional components of the vielbein and Lorentz connection that contain the time index. The physical meaning of the time component of the vielbein $e_0{}^i \to \partial_0 u^i = v^i$ is simple – this is the velocity of a point of the medium. This interpretation is natural from the physical standpoint because the motion of continuously distributed dislocations means a flowing of the medium. In fact, the liquid can be imagined as the elastic media with a continuous distribution of moving dislocations. This means that the dynamical theory of defects based on the Riemann–Cartan geometry must include hydrodynamics. It is not clear at present how this can happen. Physical interpretation of the other components of the vielbein and the Lorentz connection with the time index also remains unclear.

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Appendix. Some differential geometry

In the appendix, we briefly present the main facts from differential geometry and introduce the notation used in this review. The description is given in local coordinates, which is sufficient for our purposes. We recommend [97] as a textbook on differential geometry.

If the metric g and the affine connection Γ are given, then we say that the geometry is defined on a differentiable manifold $\mathbb{M}$, $\mathsf{dim}\,\mathbb{M}=m$. We assume that all fields on the manifold are given by smooth $\mathcal{C}^\infty(\mathbb{M})$ functions except, possibly, some singular points, and do not stipulate that in what follows. We also assume that the manifold $\mathbb{M}$ is topologically trivial, i.e., diffeomorphic to the Euclidean space $\mathbb{R}^m$.

In a local coordinate system x^μ, $\mu=1,\dots,m$, the metric is given by a nondegenerate symmetric covariant second rank tensor $g_{\mu\nu}(x)$, which defines the scalar product of vector fields $X=X^\mu\partial_\mu$, $Y=Y^\mu\partial_\mu$

$$(X,Y)=X^\mu Y^\nu g_{\mu\nu}, \qquad g_{\mu\nu}=g_{\nu\mu}, \qquad \mathsf{det}\,g_{\mu\nu}\neq 0. \tag{105}$$

In general, the scalar product may be not positive definite. If the scalar product is positive definite, we say that a Riemannian metric is given on a manifold. By definition, the metric is a covariant tensor field, i.e., it transforms under coordinate changes $x^\mu\to x^{\mu'}(x)$ by the tensor law

$$g_{\mu'\nu'}=\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}g_{\mu\nu}.$$

This means that the scalar product of vector fields (X,Y) is a scalar field. In a similar way, contractions with the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, $g^{\mu\nu}g_{\nu\rho}=\delta^\mu_\rho$, allows one to build scalar fields from higher-rank tensors or to lower their rank. The metric is also used for lowering and raising of tensor indices.

An affine connection on a manifold in a local coordinate system is given by the set of coefficients $\Gamma_{\mu\nu}{}^\rho(x)$ that transform under the diffeomorphisms as

$$\Gamma_{\mu'\nu'}{}^{\rho'}=\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\Gamma_{\mu\nu}{}^\rho\frac{\partial x^{\rho'}}{\partial x^\rho}+\frac{\partial^2 x^\rho}{\partial x^{\mu'}\partial x^{\nu'}}\frac{\partial x^{\rho'}}{\partial x^\rho}. \tag{106}$$

These coefficients do not constitute a tensor field because of the presence of an inhomogeneous term in (106). An affine connection on a manifold defines covariant derivatives of tensor fields. In particular, covariant derivatives of a vector field and 1-form $A=dx^\mu A_\mu$ have the form

$$\nabla_\mu X^\nu=\partial_\mu X^\nu+X^\rho\Gamma_{\mu\rho}{}^\nu, \tag{107}$$

$$\nabla_\mu A_\nu=\partial_\mu A_\nu-\Gamma_{\mu\nu}{}^\rho A_\rho. \tag{108}$$

The covariant derivative of a scalar field coincides with the partial derivative $\nabla_\mu\varphi=\partial_\mu\varphi$. The covariant derivative of higher-rank tensors is built in a similar way and contains one term with the plus and minus sign for each contravariant and covariant index, respectively. One can easily check that the covariant derivative of a tensor of an arbitrary type (r,s) is a tensor field of type $(r,s+1)$, i.e., it has one additional covariant index. We note that the covariant derivative of a product of tensor fields may contain arbitrary contractions of indices. For example,

$$\partial_\mu(X^\nu A_\nu)=\nabla_\mu(X^\nu A_\nu)=(\nabla_\mu X^\nu)A_\nu+X^\nu(\nabla_\mu A_\nu).$$

Because the inhomogeneous term in (106) is symmetric in indices μ' and ν' the antisymmetric part of the affine connection $2\Gamma_{[\mu\nu]}{}^{\rho}$ forms a tensor field of type $(1, 2)$, which is called the torsion tensor

$$T_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}. \tag{109}$$

In general, the connection $\Gamma_{\mu\nu}{}^{\rho}$ has no symmetry in its indices and is not related to the metric $g_{\mu\nu}$ in any way because these notions define different geometric operations on a manifold $\mathbb{M}$. We then say that the affine geometry is given on $\mathbb{M}$. We emphasize that the metric and the affine connection are defined arbitrarily and are completely independent geometric notions. Therefore, in the construction of physical models, they can be considered independent fields having different physical interpretations.

If the affine geometry is given on a manifold, then we can construct the nonmetricity tensor $Q_{\mu\nu\rho}$ that is by definition equal to the covariant derivative of the metric:

$$-Q_{\mu\nu\rho} = \nabla_{\mu} g_{\nu\rho} = \partial_{\mu} g_{\nu\rho} - \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} - \Gamma_{\mu\rho}{}^{\sigma} g_{\nu\rho}. \tag{110}$$

By construction, the nonmetricity tensor is symmetric with respect to the permutation of the last two indices: $Q_{\mu\nu\rho} = Q_{\mu\rho\nu}$. We note that we need both objects, the metric and the connection to define the nonmetricity.

Thus, for a given metric and connection we constructed two tensor fields: the torsion and the nonmetricity tensor. We prove that for a given metric, torsion, and nonmetricity tensor we can uniquely reconstruct the corresponding affine connection. Equation (110) can always be solved for the connection Γ. Indeed, the linear combination

$$\nabla_{\mu} g_{\nu\rho} + \nabla_{\nu} g_{\rho\mu} - \nabla_{\rho} g_{\mu\nu}$$

yields the expression for the affine connection with all lowered indices:

$$\begin{aligned} \Gamma_{\mu\nu\rho} = \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} = \frac{1}{2}(\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}) + \frac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}) \\ + \frac{1}{2}(Q_{\mu\nu\rho} + Q_{\nu\rho\mu} - Q_{\rho\mu\nu}). \end{aligned} \tag{111}$$

The right hand side of this equality is symmetric in the indices μ and ν except one term $T_{\mu\nu\rho}/2$, and this is in accord with the definition of the torsion tensor (109). Thus, to define the affine geometry on a manifold $\mathbb{M}$, it is necessary and sufficient to define three tensor fields: metric, torsion, and nonmetricity. We stress once again that all three object can be specified in a completely arbitrary way, and they can be considered different dynamical variables in models of mathematical physics.

It is easy to compute the number of independent components of connection, torsion, and nonmetricity tensors:

$$[\Gamma_{\mu\nu}{}^{\gamma}] = m^3, \qquad [T_{\mu\nu}{}^{\rho}] = \frac{m^2(m-1)}{2}, \qquad [Q_{\mu\nu\rho}] = \frac{m^2(m+1)}{2}.$$

This shows that the total number of independent components of the torsion and nonmetricity tensors equals the number of components of the affine connection.

We now consider particular cases of the affine geometry.

In the attempt to unite gravity and electromagnetism, H. Weyl considered the nonmetricity tensor of a special type [98]

$$Q_{\mu\nu\rho} = W_\mu g_{\nu\rho}, \tag{112}$$

where W_μ is the Weyl form identified with the electromagnetic potential (here, torsion was assumed to be identically equal to zero). We say that the *Riemann–Cartan–Weyl geometry* is defined on a manifold if there are given a metric, torsion, and nonmetricity of special type (112).

If the nonmetricity tensor is identically zero, $Q_{\mu\nu\rho} = 0$, but the metric and torsion are nontrivial, then we say that the *Riemann–Cartan geometry* is given on a manifold. As a consequence of (111), the affine connection is defined uniquely by the metric and torsion in this case:

$$\Gamma_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\mu\nu}). \tag{113}$$

This connection is called metrical because the covariant derivative of the metric is identically equal to zero:

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^\sigma g_{\sigma\rho} - \Gamma_{\mu\rho}{}^\sigma g_{\nu\sigma} = 0. \tag{114}$$

The metricity condition provides commutativity of covariant differentiation and the raising and lowering of indices.

If the torsion tensor identically vanishes, $T_{\mu\nu}{}^\rho = 0$, and nonmetricity have special form (112), then we say that the *Riemann–Weyl geometry* is given.

If both nonmetricity and torsion tensors are identically equal to zero, $Q_{\mu\nu\rho} = 0$, $T_{\mu\nu\rho} = 0$, and the metric is nontrivial, then we say that *Riemannian geometry* is given on a manifold. In this case, the metrical connection is symmetric with respect to the first two indices and uniquely defined by the metric

$$\widetilde{\Gamma}_{\mu\nu\rho} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \tag{115}$$

This connection is called the *Levy-Civita connection* or *Christoffel symbols*.

We use the tilde to denote the geometrical objects constructed only for the metric and zero torsion and nonmetricity, i.e., in a (pseudo-)Riemannian geometry. The prefix pseudo- is used when the metric is not positive definite.

From the expression for Christoffel symbols (115), we see that

$$\partial_\mu g_{\nu\rho} = \widetilde{\Gamma}_{\mu\nu\rho} + \widetilde{\Gamma}_{\mu\rho\nu}. \tag{116}$$

This means that for the Christoffel symbols to be zero in some coordinate system it is necessary and sufficient that the metric components be constant in these coordinates. In another coordinate system, they may be nontrivial because Christoffel symbols are not components of a tensor. For example, the Christoffel symbols for Euclidean space are zero in the Cartesian coordinates but differ from zero, e.g., in spherical or cylindrical coordinate systems.

In the case where the nonmetricity tensor and torsion are identically zero, and there is a coordinate system in which the metric is equal to the diagonal unit matrix in the neighborhood of every point, and, consequently, the Christoffel symbols are zero, the geometry is called locally Euclidean. The corresponding coordinate system is called *Cartesian*.

We say that a vector field $X = X^\mu\partial_\mu$ is transported parallel along a curve $x^\mu(t)$, $t \in \mathbb{R}$, if

$$\dot{x}^\nu \nabla_\nu X^\mu = 0,$$

where $\dot{x}^\mu = dx^\mu/dt$ is the tangent vector to the curve. Multiplying this equation by dt we obtain

$$\delta X^\mu = \delta x^\nu \partial_\nu X^\mu = -\delta x^\nu \Gamma_{\nu\rho}{}^\mu X^\rho,$$

where $\delta x^\nu = dt\dot{x}^\nu$. We say that under the parallel transport from a point x^ν to a neighboring point $x^\nu + \delta x^\nu$, the vector X^μ acquires a differential δX^μ, which is linear in δx^ν and the components of the vector field. In a similar way, we define parallel transport of arbitrary tensor fields along a curve $x^\mu(t)$. The result of parallel transport between two points depends in general on a curve connecting these points. We note that parallel transport along a curve is defined only by an affine connection and has no relation to a metric.

There are two types of distinguished curves $x^\mu(t)$ in an affine geometry: geodesics and extremals. A geodesic line is a curve such that a vector tangent to it remains tangent under parallel transport along the curve. With the parameter t along the curve chosen canonically, a geodesic line is defined by the system of ordinary nonlinear differential equations

$$\ddot{x}^\mu = -\Gamma_{\nu\rho}{}^\mu \dot{x}^\nu \dot{x}^\rho. \tag{117}$$

Although a geodesic line is defined only by the symmetric part of the affine connection $\Gamma_{\{\mu\nu\}}{}^\rho$, the latter depends nontrivially on torsion and nonmetricity. Explicit expression (111) yields

$$\Gamma_{\{\mu\nu\}}{}^\rho = \widetilde{\Gamma}_{\mu\nu}{}^\rho + \frac{1}{2}(T^\rho{}_{\mu\nu} + T^\rho{}_{\nu\mu}) + \frac{1}{2}(Q_{\mu\nu}{}^\rho + Q_{\nu\mu}{}^\rho - Q^\rho{}_{\mu\nu}).$$

The second type of distinguished curves in an affine geometry are extremals or lines of extremal length connecting two points. These lines are exclusively defined by the metric. With the parameter along the curve chosen canonically, extremals are defined by an equation similar to (117),

$$\ddot{x}^\mu = -\widetilde{\Gamma}_{\nu\rho}{}^\mu \dot{x}^\nu \dot{x}^\rho. \tag{118}$$

However, we now have not a general affine connection on the right-hand side but Christoffel symbols constructed only from the metric. We see that geodesics and extremals are in general different curves on a manifold. In a Riemann–Cartan geometry, geodesics and extremals coincide if and only if the torsion tensor is antisymmetric in all three indices. In Riemannian geometry, geodesics and extremals always coincide.

The primary role in differential geometry is played by the curvature tensor of the affine connection, which arises in different contexts. In local coordinates, it is defined by the expression

$$R_{\mu\nu\rho}{}^\sigma = \partial_\mu \Gamma_{\nu\rho}{}^\sigma - \Gamma_{\mu\rho}{}^\lambda \Gamma_{\nu\lambda}{}^\sigma - (\mu \leftrightarrow \nu), \tag{119}$$

where the parenthesis $(\mu \leftrightarrow \nu)$ denote the previous terms with the permuted indices μ and ν. It can be easily verified that curvature (119) is indeed a tensor field. We note that the curvature tensor has no relation to a metric and is defined entirely by the connection. In an affine geometry, the curvature tensor with all of its indices lowered has no symmetry under permutations of indices except antisymmetry in the first two indices.

Contraction of the curvature tensor in the two indices yields the Ricci tensor $R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho}$, which is also defined entirely by the connection. In Riemannian geometry, the Ricci tensor for the Levi-Civita connection is symmetric with respect to the permutation of indices: $\widetilde{R}_{\mu\nu} = \widetilde{R}_{\nu\mu}$. In a Riemann–Cartan geometry, this symmetry is generally absent for a nonzero torsion tensor.

If a metric is also given on a manifold, then we can construct the scalar curvature $R = R_{\mu\nu} g^{\mu\nu}$.

The affine connection is called locally trivial if in the neighborhood of every point one can choose the coordinate system in which all components of the connection are zero. Now we formulate two important theorems.

Theorem 1. *For the local triviality of the affine connection it is necessary and sufficient that its torsion and curvature tensors are equal to zero on* $\mathbb{M}$.

The proof of this theorem is reduced to an analysis of the transformation rule for the affine connection components in (106). If in the new coordinate system the symmetric part of the connection components is equal to zero, then the system of differential equations for the transition functions appears. The local integrability condition for this system of equations is provided by the equality of the curvature tensor to zero.

A more thorough statement is proven in [99].

Theorem 2. *If for a given affine geometry the torsion, nonmetricity, and curvature tensors are equal to zero on the whole manifold, then this manifold is isometric either to the whole (pseudo-)Euclidean space* $\mathbb{R}^m$ *or to the quotient space* $\mathbb{R}^m/\mathbb{G}$*, where* $\mathbb{G}$ *is a discreet transformation group acting freely.*

In the last theorem, we suppose that both the metric and affine connection are given on a manifold.

The Riemann–Cartan geometry defined by the metric and torsion provides the basis for the geometric theory of defects. In the analysis of such models, it is more convenient to introduce the Cartan variables: the vielbein and $\mathbb{SO}(m)$ connection instead of the metric and torsion. We assume here that a metric on a manifold is positive definite. If a metric were not positively definite then the $\mathbb{SO}(p,q)$ connection would appear, where $p+q=m$.

For a Riemannian metric, the orthonormal vielbein $e_\mu{}^i(x)$, $i = 1, \ldots, m$, is defined by the system of quadratic equations

$$g_{\mu\nu} = e_\mu{}^i e_\nu{}^j \delta_{ij}, \tag{120}$$

where δ_{ij} is the Kronecker symbol. This relation uniquely defines metric for a given vielbein. Conversely, system of equations (120) defines the vielbein for a given metric up to local $\mathbb{SO}(m)$ rotations. We distinguish Greek and Latin indices because different transformation groups act on them. From definition (120), we see that $\mathsf{det}\, e_\mu{}^i \neq 0$.

Components of the inverse vielbein $e^\mu{}_i$, $e^\mu{}_i e_\nu{}^i = \delta^\mu_\nu$, can be considered components of m orthonormal vector fields $e_i = e^\mu{}_i \partial_\mu$ on a manifold $\mathbb{M}$ with respect to the metric g

$$(e_i, e_j) = e^\mu{}_i e^\nu{}_j g_{\mu\nu} = \delta_{ij}.$$

Components of the vielbein $e_\mu{}^i$ define m orthonormal 1-forms $e^i = dx^\mu e_\mu{}^i$ on a manifold.

It is known that any manifold $\mathbb{M}$ can be equipped with a Riemannian metric (see, for e.g., [100]). At the same time, the global existence of a vielbein provides, in particular, an orientation of the manifold $\mathbb{M}$. Thus, a vielbein may exist globally only on orientable manifolds. There are also other topological restrictions for the global existence of a vielbein that we do not discuss here.

The coordinate basis ∂_μ of the tangent space in every point of $\mathbb{M}$ is called holonomic. Components of tensors of an arbitrary type may also be considered with respect to the unholonomic basis e_i of tangent spaces defined by the vielbein. For example, a vector field has the components

$$X = X^\mu \partial_\mu = X^i e_i, \qquad X^i = X^\mu e_\mu{}^i.$$

We always assume that the transformation of Greek indices into Latin ones and vice versa is performed with the help of the vielbein.

We now define the $\mathbb{SO}(m)$ connection $\omega_{\mu i}{}^j(x)$. This is the name of the connection of the principal fiber bundle with the structure Lie group $\mathbb{SO}(m)$ and the base $\mathbb{M}$. If the Riemann-Cartan geometry is given on $\mathbb{M}$, and the vielbein is defined, then we can define the $\mathbb{SO}(m)$ connection by the relation

$$\nabla_\mu e_\nu{}^i = \partial_\mu e_\nu{}^i - \Gamma_{\mu\nu}{}^\rho e_\rho{}^i + e_\nu{}^j \omega_{\mu j}{}^i = 0. \tag{121}$$

We see that under coordinate changes, the components of $\omega_{\mu j}{}^i$ transform as a covector field in the Greek index. Thus, they define a 1-form on $\mathbb{M}$. For a given vielbein, this relation provides a one-to-one correspondence between components of the connections $\Gamma_{\mu\nu}{}^\rho$ and $\omega_{\mu j}{}^i$. The $\mathbb{SO}(m)$ connection defines covariant derivatives for components of tensor fields relative to an unholonomic basis. For example,

$$\nabla_\mu X^i = \partial_\mu X^i + X^j \omega_{\mu j}{}^i, \qquad \nabla_\mu X_i = \partial_\mu X_i - \omega_{\mu i}{}^j X_j. \tag{122}$$

The covariant derivative is naturally defined for a tensor field having both Greek and Latin indices. Taking the covariant derivative of Eqn (120) leads to the antisymmetry of the components $\omega_\mu{}^{ij} = -\omega_\mu{}^{ji}$. This means that the 1-form $dx^\mu \omega_{\mu j}{}^i$ takes values in the Lie algebra $\mathfrak{so}(m)$, and this indeed corresponds to the $\mathbb{SO}(m)$ connection.

In the general case of an affine geometry, the Cartan variables can also be defined by relations (120) and (121). For this, we have to replace the Kronecker symbol on the right-hand side of (120) with an arbitrary nondegenerate symmetric matrix η_{ij}. In this case, relation (121) defines a linear $\mathbb{GL}(m,\mathbb{R})$ connection. In the Riemann–Cartan geometry with the Lorentzian-signature metric, we would have the Lorentz $\mathbb{SO}(1,m-1)$ connection.

We consider local rotations with a matrix $S_j{}^i(x) \in \mathbb{SO}(m)$ to show that the components $\omega_{\mu j}{}^i$ do define an $\mathbb{SO}(m)$ connection in the Riemann–Cartan geometry. By definition, the components of vector fields and 1-forms transform under local rotations according to the rule

$$X'^i = X^j S_j{}^i, \qquad X'_i = S^{-1}{}_i{}^j X_j. \tag{123}$$

In order that covariant derivatives (122) have the tensor transformation rule under local rotations, it is necessary and sufficient that the components of the $\mathbb{SO}(m)$ connection transform according to the rule

$$\omega'_{\mu i}{}^{j} = S^{-1}{}_{i}{}^{k}\omega_{\mu k}{}^{l}S_{l}{}^{j} + \partial_\mu S^{-1}{}_{i}{}^{k}S_{k}{}^{j}. \tag{124}$$

This is the transformation law for the $\mathbb{SO}(m)$ connection, indeed. The same transformation law follows from definition (121) if the vielbein $e_\mu{}^i$ transforms as a vector with respect to the index i, and the Christoffel symbols remain unchanged.

Of course, we can introduce the metrical and $\mathbb{SO}(m)$ connections on $\mathbb{M}$ in an independent way. If we require afterwards that the $\mathbb{SO}(m)$ connection acts on tensor field components relative to an unholonomic basis defined by the vielbein, we obtain a one-to-one correspondence between the components of the connections (121).

We now express the components of the metric connection $\Gamma_{\mu\nu}{}^\rho$ through the vielbein $e_\mu{}^i$ and $\mathbb{SO}(m)$ connection $\omega_{\mu j}{}^i$ with the help of relation (121) and substitute them into the definitions of torsion (109) and curvature (119). As a result, we obtain expressions for torsion and curvature in terms of the Cartan variables

$$T_{\mu\nu}{}^i = \partial_\mu e_\nu{}^i - e_\mu{}^j\omega_{\nu j}{}^i - (\mu \leftrightarrow \nu), \tag{125}$$

$$R_{\mu\nu j}{}^i = \partial_\mu \omega_{\nu j}{}^i - \omega_{\mu j}{}^k\omega_{\nu k}{}^i - (\mu \leftrightarrow \nu), \tag{126}$$

where

$$T_{\mu\nu}{}^i = T_{\mu\nu}{}^\rho e_\rho{}^i, \qquad R_{\mu\nu j}{}^i = R_{\mu\nu\rho}{}^\sigma e^\rho{}_j e_\sigma{}^i.$$

Torsion (125) and curvature (126) are 2-forms on the manifold $\mathbb{M}$ with the values in the vector space and Lie algebra $\mathfrak{so}(m)$, respectively.

An $\mathbb{SO}(m)$ connection $\omega_{\mu j}{}^i$ is called locally trivial if every point has a neighborhood containing this point such that it has the form

$$\omega_{\mu j}{}^i = \partial_\mu S^{-1}{}_j{}^k S_k{}^i. \tag{127}$$

Obviously, after local rotation with the matrix $S^{-1}{}_j{}^i$, all components of the connection become zero. This connection is also called a pure gauge. One can easily verify that the curvature of a locally trivial connection identically vanishes. The inverse statement is also valid.

Theorem 3. *An $\mathbb{SO}(m)$ connection is locally trivial if and only if its curvature form is identically zero on $\mathbb{M}$.*

This theorem is valid for any structure Lie group. The proof is reduced to an analysis of transformation rule (124). If the left-hand side of this equation is zero, then we have a system of equations for the field $S_j{}^i(x)$ for which vanishing of curvature tensor is the local integrability condition.

The space with zero curvature tensor $R_{\mu\nu j}{}^i = 0$ is called the space of absolute parallelism because the parallel displacement of a vector does not depend on the path connecting two fixed points of the manifold.

We can perform a local $\mathbb{SO}(m)$ rotation for the locally trivial $\mathbb{SO}(m)$ connection such that it becomes zero: $\omega_{\mu j}{}^i = 0$. Then, the zero-torsion equality becomes

$$\partial_\mu e_\nu{}^i - \partial_\nu e_\mu{}^i = 0.$$

This equation is the local integrability condition for the system of equations

$$\partial_\mu y^i = e_\mu{}^i \tag{128}$$

for m functions $y^i(x)$. A solution of this system of equations yields the transition functions to Cartesian coordinates. Thus, the equalities of curvature and torsion tensors to zero are the necessary and sufficient conditions for the existence of the fields $S_j{}^i(x)$ and $y^i(x)$, i.e., the existence of such a local rotation and coordinate system where the connection vanishes and the metric becomes Euclidean. We note that equality of torsion tensor to zero alone is not enough for the existence of a Cartesian coordinate system.

Three-dimensional space is considered in the geometric theory of defects. We make two remarks concerning this. In lower dimensions, the algebraic structure of the curvature tensor with all lowered indexes becomes much simpler. In two dimensions the full curvature tensor in the Riemann–Cartan geometry is in the one-to-one correspondence with its scalar curvature:

$$R_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})R.$$

In three dimensional space, the full curvature tensor is in the one-to-one correspondence with its Ricci tensor

$$R_{ijkl} = \delta_{ik}R_{jl} - \delta_{il}R_{jk} - \delta_{jk}R_{il} + \delta_{jl}R_{ik} - \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})R. \tag{129}$$

These formulas are also correct for nonzero torsion.

Concluding this appendix, we write the identity that is valid in the Riemann–Cartan geometry in an arbitrary number of dimensions:

$$R(e,\omega) + \frac{1}{4}T_{ijk}T^{ijk} - \frac{1}{2}T_{ijk}T^{kij} - T_iT^i - \frac{2}{e}\partial_\mu(eT^\mu) = \widetilde{R}(e), \qquad e = \mathsf{det}\, e_\mu{}^i. \tag{130}$$

The Riemannian scalar curvature on the right-hand side of this equality is constructed only from the vielbein for zero torsion.

References

[1] K. Kondo. On the geometrical and physical foundations of the theory of yielding. In *Proc. 2nd Japan Nat. Congr. Applied Mechanics*, pages 41–47, Tokyo, 1952.

[2] J. F. Nye. Some geometrical relations in dislocated media. *Acta Metallurgica*, 1:153, 1953.

[3] B. A. Bilby, R. Bullough, and E. Smith. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. *Proc. Roy. Soc. London*, A231:263–273, 1955.

[4] E. Kröner. *Kontinums Theories der Versetzungen und Eigenspanungen.* Spriger–Verlag, Berlin – Heidelberg, 1958.

[5] L. I. Sedov and V. L. Berditchevski. A dynamical theory of dislocations. In E. Kröner, editor, *Mechanics of Generalized Continua, UITAM symposium*, pages 214–238, 1967.

[6] M. Kléman. The general theory of dislocations. In Nabarro F. R. N., editor, *Dislocations In Solids, Vol. 5*, pages 243–297, Amsterdam, 1980. North-Holland Publishing Company.

[7] E. Kröner. Continuum theory of defects. In R. Balian et al., editor, *Less Houches, Session XXXV, 1980 – Physics of Defects*, pages 282–315. North-Holland Publishing Company, 1981.

[8] I. E. Dzyaloshinskii and G. E. Volovik. *Ann. Phys.*, 125:67, 1988.

[9] A. Kadić and D. G. B. Edelen. *A gauge theory of dislocations and disclinations.* Springer–Verlag, Berlin – Heidelberg, 1983.

[10] I. A. Kunin and B. I. Kunin. Gauge theories in mechanics. In *Trends in Application of Pure Mathematics to Mechanics. Lecture Notes in Physics, V.249.*, pages 246–249, Berlin – Heidelberg, 1986. Springer–Verlag.

[11] H. Kleinert. *Gauge fields in condenced matter*, volume 2. World Scientific, Singapore, 1990.

[12] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman. Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilaton invariance. *Phys. Rep.*, 258(1&2):1–171, 1995.

[13] E. Cartan. Sur une generalisation de la notion de courburu de Riemann et les aspases a torsion. *Compt. Rend. Acad. Sci. (Paris)*, 174:593–595, 1922.

[14] C. Malyshev. The $T(3)$-gauge model, the Einstein-like gauge equation, and Volterra dislocations with modified asymptotics. *Ann. Phys.*, 286:249–277, 2000.

[15] M. Lazar. Dislocation theory as a 3-dimensional translation gauge theory. *Ann. Phys. (Leipzig)*, 9:461–473, 2000.

[16] M. Lazar. An elastoplastic theory of dislocations as a physical field theory with torsion. *J. Phys. A*, 35:1983–2004, 2002.

[17] M. Lazar. A nonsingular solution of the edge dislocation in the gauge theory of dislocations. *J. Phys. A*, 35:1983–2004, 2002.

[18] F. C. Frank. On the theory of liquid crystals. *Discussions Farad. Soc.*, 25:19–28, 1958.

[19] I. E. Dzyaloshinskii and G. E. Volovik. On the concept of local invariance in the theory of spin glasses. *J. Physique*, 39(6):693–700, 1978.

[20] J. A. Hertz. Gauge model for spin-glasses. *Phys. Rev.*, B18(9):4875–4885, 1978.

[21] N. Rivier and D. M. Duffy. Line defects and tunneling modes in glasses. *J. Physique*, 43(2):293–306, 1982.

[22] M. O. Katanaev and I. V. Volovich. Theory of defects in solids and three-dimensional gravity. *Ann. Phys.*, 216(1):1–28, 1992.

[23] M. O. Katanaev. Wedge dislocation in the geometric theory of defects. *Theor. Math. Phys.*, 135(2):733–744, 2003.

[24] M. O. Katanaev. One-dimensional topologically nontrivial solutions in the Skyrme model. *Theor. Math. Phys.*, 138(2):163–176, 2004.

[25] C. Furtado and F. Moraes. On the binding of electrons and holes to disclinations. *Phys. Lett.*, A188:394–396, 1994.

[26] C. Furtado, B. G. C. da Cunha, F. Moraes, E. R. Bezerra de Mello, and V. B. Bezzerra. Landau levels in the presence of disclinations. *Phys. Lett.*, A195:90–94, 1994.

[27] F. Moraes. Enhancement of the magnetic moment of the electron due to a topological defect. *Mod. Phys. Lett.*, A10(31):2335–2338, 1995.

[28] S. Azevedo and F. Moraes. Topological Aharonov–Bohm effect around a disclination. *Phys. Lett. A*, 246:374–376, 1998.

[29] A. P. Balachandran, V. John, A. Momen, and F. Moraes. Anomalous defects and their quantized transverse conductivities. *Int. J. Mod. Phys. A*, 13(5):841–861, 1998.

[30] C. Furtado and F. Moraes. Landau levels in the presence of a screw dislocation. *Europhys. Lett.*, 45(3):279–282, 1999.

[31] R. Bausch, R. Schmitz, and L. A. Turski. Single-particle quantum states in a crystal with topological defects. *Phys. Rev. Lett.*, 80(11):2257–2260, 1998.

[32] R. Bausch, R. Schmitz, and L. A. Turski. Quantum motion of electrons in topologically distorted crystal. *Ann. Phys. (Leipzig)*, 8(3):181–189, 1999.

[33] C. Furtado and F. Moraes. Harmonic oscillator interacting with conical singularities. *J. Phys. A*, 33:5513–5519, 2000.

[34] S. Azevedo and F. Moraes. Two-dimensional scattering by disclinations in monolayer graphite. *J. Phys. C*, 12:7421–7424, 2000.

[35] S. Azevedo and J. Pereira. Double Aharonov–Bohm effect in a medium with a disclination. *Phys. Lett. A*, 275:463–466, 2000.

[36] C. Furtado, V. B. Bezerra, and F. Moraes. Berry's quantum phase in media with dislocations. *Europhys. Lett.*, 52(1):1–7, 2000.

[37] C. A. de Lima Ribeiro, C. Furtado, and F. Moraes. On the localization of electrons and holes by a disclination core. *Phys. Lett. A*, 288:329–334, 2001.

[38] S. R. Vieira and S. Azevedo. Double Aharonov–Bohm effect in a medium with a linear topological defect. *Phys. Lett.*, 288:29–32, 2001.

[39] S. Azevedo. A charged particle with magnetic moment in a medium with a disclination. *J. Phys. A*, 34:6081–6085, 2001.

[40] C. Furtado, V. B. Bezerra, and F. Moraes. Quantum scattering by a magnetic flux screw dislocation. *Phys. Lett.*, A289:160–166, 2001.

[41] G. de A Marques, C. Furtado, V. B. Bezerra, and F. Moraes. Landau levels in the presence of topological defects. *J. Phys. A*, 34:5945–5954, 2001.

[42] C. Furtado, C. A. de Lima Ribeiro, and S. Azevedo. Aharonov–Bohm effect in the presence of a density of defects. *Phys. Lett.*, A296:171–175, 2002.

[43] S. Azevedo. Topological Aharonov–Bohm effect in a two-dimensional harmonic oscillator. *Phys. Lett.*, A293:283–286, 2002.

[44] S. Azevedo. Bound particle with magnetic moment in a space with topological defect. *Int. J. Quantum Chem.*, 90:1596–1599, 2002.

[45] S. Azevedo. Bound charge moving in a magnetic field in a space with a topological defect. *Mod. Phys. Lett.*, A17:1263–1268, 2002.

[46] S. Azevedo. Charged particle with magnetic moment in the background of line topological defect. *Phys. Lett.*, A307:65–68, 2003.

[47] A. De S. Barbosa, E. R. Bezerra De Mello, and V. B. Bezerra. Coulomb interaction between two charged particles on a cone. *Int. J. Mod. Phys. A*, 14(22):3565–3580, 1999.

[48] S. Azevedo, C. Furtado, and F. Moraes. Charge localization around disclinations in monolayer graphite. *phys. stat. sol. (b)*, 207:387–392, 1998.

[49] F. Moraes. Casimir effect around disclination. *Phys. Lett.*, A204:399–404, 1995.

[50] E. R. Bezerra de Mello, V. B. Bezerra, and Yu. V. Grats. Self-forces in the spacetime of multiple cosmic strings. *Class. Quantum Grav.*, 15:1915–1925, 1998.

[51] S. Azevedo. Harmonic oscillator in a space with a linear topological defect. *Phys. Lett.*, A288:33–36, 2001.

[52] C. Furtado, V. B. Bezerra, and F. Moraes. Aharonov–Bohm effect for bound states in Kaluza–Klein theory. *Mod. Phys. Lett.*, A15(4):253–258, 2000.

[53] I. Tanaka. Geometrical aspect of pinning in superconducting material. *Phys. Lett. A*, 277:262–266, 2000.

[54] V. A. De Lorenci and E. S. Jr. Moreira. Classical self-forces in a space with a topological defect. *Phys. Rev.*, D65:085013, 2002.

[55] S. Azevedo and F. Moraes. Self-force on a point charge and linear source in the space of a screw dislocation. *Phys. Lett.*, 267:208–211, 2000.

[56] J. A. Miranda and F. Moraes. Geometric approach to viscous fingering on a cone. *J. Phys. A*, 36:863–874, 2003.

[57] F. Moraes. Geodesics around a dislocation. *Phys. Lett.*, A214(3,4):189–192, 1996.

[58] A. de Padua, F. Parisio-Filho, and F. Moraes. Geodesics around line defects in elastic solids. *Phys. Lett. A*, 238:153–158, 1998.

[59] M. O. Katanaev and I. V. Volovich. Scattering on dislocations and cosmic strings in the geometric theory of defects. *Ann. Phys.*, 271:203–232, 1999.

[60] M. F. Miri and N. Rivier. Continuum elasticity with topological defects, including dislocations and extra-matter. *J. Phys. A*, 35:1727–1739, 2002.

[61] L. D. Landau and E. M. Lifshits. *Theory of Elasticity*. Pergamon, Oxford, 1970.

[62] A. M. Kosevich. *Physical mechanics of real crystals*. Naukova dumka, Kiev, 1981. [in Russian].

[63] A. Verçin. Metric–torsion gauge theory of continuum line defects. *Int. J. Theor. Phys.*, 29(1):7–21, 1990.

[64] V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskii. *The Inverse Scattering Method*. Nauka, Moscow, 1980. [in Russian]; English transl.: S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov *The Inverse Scattering Method* Plenum, New York (1984).

[65] R. Rajaraman. *Solitons and Instantons in Quantum Field Theory*. North–Holland, Amsterdam, 1982.

[66] L. A. Takhtadzhyan and L. D. Faddeev. *The Hamiltonian Methods in the Theory of Solitons*. Nauka, Moscow, 1986. [in Russian]; English transl.: L. D. Faddeev and L. A. Takhtajan *The Hamiltonian Methods in the Theory of solitons,* Berlin, Springer (1987).

[67] W. J. Zakrzewski. *Low Dimensional Sigma Models*. Adam Hilger, Bristol – Philadelphia, 1989.

[68] Yu. P. Rybakov and V. I. Sanyuk. *Multidimensional Solitons*. Russian Univ. of People's Friendship Publ., Moscow, 2001. [in Russian].

[69] T. H. R. Skyrme. Nonlinear field theory. *Proc. Roy. Soc. London*, A260:127–138, 1961.

[70] L. D. Landau and E. M. Lifshits. *Electrodynamics of continuous media*. Nauka, Moscow, second edition, 1982. [in Russian].

[71] L. D. Faddeev. In search of multidimensional solitons. In *Nonlocal. Nonlinear, and Nonrenormalizable Field Theories*, pages 207–223, Dubna, 1977. Joint Inst. Nucl. Res. [in Russian].

[72] E. Cosserat and F. Cosserat. *Théorie des corps déformables*. Hermann, Paris, 1909.

[73] W. Nowacki. *Theory of asymmetric elasticity*. Pergamon Press, Oxford, 1985.

[74] N. Sandru. On some problems of the linear theory of the asymmetric elasticity. *Int. J. Eng. Sci.*, 4(1), 1966.

[75] A. Staruszkiewicz. Gravitational theory in three-dimensional space. *Acta Phys. Polon.*, 24(6(12)):735–740, 1963.

[76] G. Clement. Field–theoretic particles in two space dimensions. *Nucl. Phys. B*, 114:437–448, 1976.

[77] S. Deser, R. Jackiw, and G. 't Hooft. Three-dimensional Einstein gravity: Dynamics of flat space. *Ann. Phys.*, 152(1):220–235, 1984.

[78] A. Holz. Geometry and action of arrays of disclinations in crystals and relation to (2+1)-dimensional gravitation. *Class. Quantum Grav.*, 5:1259–1282, 1988.

[79] C. Kohler. Point particles in 2+1-dimensional gravity as defects in solid continua. *Class. Quantum Grav.*, 12:L11–L15, 1995.

[80] C. Kohler. Line defects in solid continua and point particles in $(2+1)$-dimensional gravity. *Class. Quantum Grav.*, 12:2977–2993, 1995.

[81] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Pergamon, New York, 1962.

[82] V. Guillemin and S. Sternberg. *Geometric Asymptotics*. American Mathematical Society, Providence, Phode Island, 1977.

[83] E. Janke, F. Emde, and F. Lösch. *Tafeln Höherer Funktionen*. B. G. Teubner Verlagsgesellschaft, Stuttgart, sechste edition, 1960.

[84] Y. Aharonov and D. Bohm. Significance of electromagnetic potentials in the quantum theory. *Phys. Rev.*, 115(3):485–491, 1959.

[85] E. L. Feinberg. On a "special role" of electromagnetic potentials in quantum mechanics. *UFN*, 78(No.1):53–64, 1962. [in Russian.]

[86] V. D. Skarzhinskii. The Aharonov–Bohm effect: theoretical calculations and interpolation. *FIAN Proc.*, 167:139–161, 1986. [in Russian.]

[87] M. O. Katanaev. Euclidean two-dimensional gravity with torsion. *J. Math. Phys.*, 38(3):946–980, 1997.

[88] I. V. Volovich and M. O. Katanaev. Quantum strings with a dynamical geometry. *JETP Lett.*, 43(5):267–269, 1986.

[89] M. O. Katanaev and I. V. Volovich. String model with dynamical geometry and torsion. *Phys. Lett.*, 175B(4):413–416, 1986.

[90] M. O. Katanaev and I. V. Volovich. Two-dimensional gravity with dynamical torsion and strings. *Ann. Phys.*, 197(1):1–32, 1990.

[91] M. O. Katanaev. New integrable model – two-dimensional gravity with dynamical torsion. *Sov. Phys. Dokl.*, 34(3):982–984, 1989.

[92] M. O. Katanaev. Complete integrability of two-dimensional gravity with dynamical torsion. *J. Math. Phys.*, 31(4):882–891, 1990.

[93] M. O. Katanaev. Conformal invariance, extremals, and geodesics in two-dimensional gravity with torsion. *J. Math. Phys.*, 32(9):2483–2496, 1991.

[94] M. O. Katanaev. All universal coverings of two-dimensional gravity with torsion. *J. Math. Phys.*, 34(2):700–736, 1993.

[95] A. Vilenkin and E. Shellard. *Cosmic Strings and Other Topological Defects*. Cambridge University Press, Cambridge, 1994.

[96] M. B. Hindmarsh and T. W. B. Kibble. Cosmic strings. *Rep. Prog. Phys.*, 58:477, 1995.

[97] B. A. Dubrovin, S. P. Novikov, A. T. Fomenko. *Modern geometry: Methods and Applications*. Nauka, Moscow, 1998, fourth edition. [In Russian]; English transl. prev. ed.: B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov *Modern Geometry: Methods and Applications,* Part 1, *The Geometry of Surfaces, Transformation Groups, and Fields,* Springer, New York (1992).

[98] H. Weyl. Gravitation und Elektrizität. *Sitz. Preuss. Akad. Wiss.*, page 465, 1918.

[99] J. A. Wolf. *Spaces of constant curvature*. University of California, Berkley, California, 1972.

[100] S. S. Chern, W. H. Chen, and K. S. Lam. *Lectures on Differential Geometry*. World Scientific, Singapore, 2000.